Enacting mathematical content knowledge in the classroom: The preservice teacher experience of lower secondary algebra

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B. Ed. (Hons), M. Ed.
in November, 2015

In fulfilment of the requirements for the
Doctor of Philosophy
In the College of Arts, Society, and Education at James Cook University

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Statement of the contribution of others

I acknowledge the intellectual support of my primary supervisor, Dr Jo Balatti, who provided feedback and advice on my research design, conceptual development, and data analysis. Jo also provided editorial assistance with this thesis. I sincerely thank Jo for generously sharing her time and energy with me.

I was awarded an Australian Postgraduate Award (APA) scholarship and received a fortnightly stipend as a living allowance. I also received financial assistance from the College of Arts, Society and Education in accordance with the *James Cook University Minimum Resources Policy* which enabled me to collect data from schools across North Queensland. I have not sought or received any other contributions.

1st November, 2015

Leah Daniel Date

Declaration of ethics

The research presented and reported in this thesis was conducted within the guidelines for research ethics outlined in the National Statement on Ethical conduct in Human Research (1999), the Joint NHMRC/AVCC Statement and Guidelines on Research Practice (1997), the James Cook University Policy on Experimentation Ethics, Standard Practices and Guidelines (2001), and the James Cook University Statement and Guidelines on Research Practice (2001). The proposed research methodology received ethics clearance from the James Cook University Ethics Review Committee (approval number H4495).

Acknowledgements

I've heard that it takes a village to raise a child, and perhaps the same can be said of a PhD student. The people in my PhD village were simply marvellous. I cannot thank them enough for their ongoing support over many years that made this experience such an enjoyable and valuable one for me.

I sincerely thank the participants of this study for their professionalism, their bravery, and their candour. It's never easy to admit your vulnerabilities or imperfections and yet these brave young men and women openly shared the good and the bad of their classroom experiences with me. I thank them for accommodating all of my requests and trusting me to report the inside story of their experiences. Collecting the data for this study was one of the most enjoyable aspects of the project.

I thank my colleagues at James Cook University for their encouragement and enthusiasm during my candidature. My education colleagues, most of whom have travelled the PhD road themselves, provided me with feedback, advice, and sugar-loaded treats to see me safely through to the end of the project. I am also very grateful to Assoc. Prof. Wayne Read and Dr Shaun Belward, based in the College of Science, Technology, and Engineering, for their support and advice regarding the mathematical content aspects of this study.

I cannot thank Dr Jo Balatti enough for her support before and during this study. I had never considered attempting such a large research project before Jo suggested the idea and without her encouragement, I doubt I would ever have embarked on a PhD. Jo recognised the opportunities and problems associated with my ideas long before I did but she allowed me to find my own way through the project. Her advice was always timely, her feedback was always thoughtful, and her curiosity and encouragement for the project made this experience a rewarding one. Jo is the epitome of a teacher to me and I am very grateful for all of her assistance with this project.

I thank my family and friends for their patience and encouragement over the past four years. In particular, I thank my sister, Cheree, who kept me laughing no matter how tired or stressed I felt. Finally, and most importantly, I thank my wonderful husband, Ryan, and my beautiful children, Maria and Lachlan, whose love and support means everything to me. Thank you for being so patient, so understanding, and so very entertaining. I feel very lucky to have you in my life.

Abstract

Effective secondary mathematics teachers possess particular forms of mathematical content knowledge (MCK) which they purposefully enact in the classroom. Secondary mathematics preservice teachers are in the process of developing their MCK and their instructional decision making skills regarding the MCK they teach. However, the quality of secondary mathematics preservice teachers' MCK has been found lacking, both nationally in Australia and internationally. Arguably even more problematic, is the challenge of finding an accurate measure of preservice teacher MCK. In contrast to the common "paper test" approach used, this interpretive Australian study sought to describe the nature of secondary mathematics preservice teacher MCK by investigating what they enact "live" in their teaching practice. Because enacted MCK results from a decision making process, the study also aimed to identify the influences impacting preservice teachers' consideration of goals and of the MCK chosen to achieve the retained goals. The study limited itself to the context of lower secondary algebra lessons, mainly up to and including linear equations.

Observation data (video footage, field notes, and lesson artefacts) pertaining to a total of six Year 8 and four Year 10 algebra lessons, taught by six 3rd and 4th year secondary preservice teachers, were collected during the preservice teachers' practicum phases. Within 48 hours of each lesson, a follow-up interview was completed with each preservice teacher. The semi-structured interview featured stimulated recall procedures using edited lesson footage. The interviews generated data concerning the decisions made by the preservice teachers that led to their enacting or withholding particular MCK during instruction. Teaching actions that attracted comment from the preservice teachers in the interviews were sorted into 137 "episodes", defined by the goal(s) pursued by the preservice teachers when performing those actions. The researcher coded the type and quality of MCK that manifested in each episode. Corresponding interview reflections were coded for evidence of decision making influences that impacted MCK related decisions. General lesson reflections, observation field notes, and lesson artefacts were also analysed for influencing elements. Lesson and interview data were analysed using pattern-seeking techniques and cross-variable analyses to identify the type and quality of MCK enacted and the influencing elements on MCK related decisions.

The results of the study suggest that five categories of influencing elements, referred to in the study as influences, impacted the MCK related decisions. The first influence was the practicum context, comprising the advice from the supervising teacher, information provided in term overviews, school perceptions of the mathematical ability of student cohorts, and content presented in the class textbook. The second influence was the preservice teachers' pedagogical intentions, evidenced in the goals they formed at the macro, meso, and micro levels of their

lessons. The third influence was the classroom circumstances that the preservice teachers considered as they made MCK related decisions. This influence comprised classroom events that captured the preservice teachers' attention and the instructional setting of classroom interactions, such as small group or whole of class instruction. The fourth influence was the preservice teachers' own MCK, MCK which they rarely sought to develop further when they prepared for their lessons. The fifth and final influence was the judgements that preservice teachers made about students. These judgements applied to how students develop mathematical understandings and to their mathematical needs, including exposure to appropriate mathematical content.

The MCK that the preservice teachers enacted showed a preponderance of procedural knowledge emphasising mathematical rules and automated sequences of procedural steps. Occasionally, there was evidence of a specialised knowledge of algebraic procedures needed for teaching lower secondary mathematics, including connections involving conceptual knowledge and algebraic ways of thinking. However, the preservice teachers only sporadically enacted conceptual knowledge and algebraic ways of thinking to supplement their presentation of rules, steps, and algebraic manipulations. The superficial treatment or notable absence of conceptual knowledge and algebraic ways of thinking in the majority of the preservice teachers' teaching episodes reduced the overall quality of the content delivered. A lack of verbal precision and a lack of attention to the limitations of the procedures demonstrated also characterised the MCK that manifested in the classroom.

The quality of the MCK delivered appeared to be associated with particular influences on the decisions the preservice teachers made. The preservice teachers tended to enact automated, imprecise, and contextually limited MCK when their own MCK was inadequate or when they made questionable judgements regarding the mathematical content they believed that students should be exposed to or the ways that students develop mathematical understandings. The preservice teachers enacted better quality MCK, which included connections involving conceptual knowledge, algebraic ways of thinking, and specialised knowledge of procedures, when their goals focussed on highlighting mathematical connections or on addressing student confusion. Stronger MCK was also evident when preservice teachers were responding to a particular student query rather than enacting MCK that they had planned to share before the lesson began. Finally, small group rather than whole of class instructional settings were associated with better quality MCK.

The study highlights the significance of the preservice teachers' own prior mathematical experiences, of their understanding of how students learn, and of their live classroom interactions with students on the MCK related decisions they make. Preservice teachers' most recent university mathematics experiences may lead to a compressed knowledge of secondary algebra procedures and an automated treatment of algebraic manipulations which are evident in their teaching actions. Their lack of experience with school learners causes them to make MCK related decisions based on their own past observations of mathematics teachers and learners, which are inevitably limited by the student vantage point from which they were observed. The live classroom context in which preservice teachers interact with students positively impacts the mathematical content delivered. By sharing mathematical ideas with students, preservice teachers refine their knowledge of students' mathematical needs and begin to unpack their own MCK to accommodate those needs, improving the quality of the MCK they subsequently enact as interactions unfold.

This study shows that preservice teacher MCK enacted in live classroom situations is not easily measured. Even when visible, it may not be a true indication of the MCK the preservice teacher possesses. The MCK that is enacted may indicate the mathematical knowledge they possess but it may also merely reflect the choices they have made, the quality of which are determined by the knowledge that preservice teachers bring to the decision making process. Hence, developing the preservice teacher MCK that manifests in their teaching actions requires attention not only to the MCK that preservice teachers hold but also to their evidence-based knowledge of how students learn mathematics. The findings of the study may improve the design and delivery of both the university-based component and the school-based component of secondary mathematics teacher education programs. Stronger partnerships between university and school-based educators are needed to (a) provide more opportunities for preservice teachers to develop evidence-based knowledge of how students learn mathematics, (b) privilege conceptual knowledge, algebraic ways of thinking, and associated connections to procedures in algebra, (c) explicitly highlight specific aspects of MCK, including precise use of mathematical terminology, that preservice teachers should be attending to in practicum lessons, and (d) provide opportunities beyond the practicum context for preservice teachers to be involved in MCK related interactions, ideally with secondary mathematics students.

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Chapter 1: Introduction

1.0 Introduction

Mathematical content knowledge (MCK) is a unique form of mathematical knowledge used by mathematics teachers in their practice. The quality of secondary mathematics preservice teachers' MCK has been questioned in recent years (Goos, 2013; Plummer & Peterson, 2009; Schwarz et al., 2008) but a lack of research exists investigating that MCK within the classroom context. Little is known about the mathematical knowledge that manifests in the lessons of preservice teachers within the practicum context or the live decision making process that underpins their MCK related teaching actions. The quality of preservice teachers' mathematical knowledge depends, in part, on the quality of teacher education programs. This qualitative study was prompted by the need to increase the body of knowledge pertaining to preservice teachers' enactment of MCK while teaching and the need to improve MCK related learning opportunities for secondary preservice teachers offered in teacher education courses. If teacher educators are to succeed in the ambitious task of developing secondary preservice teachers' knowledge of mathematics and their successful application of this knowledge in a live classroom context, a more comprehensive understanding of preservice teachers' use of their MCK during instruction is needed.

This study explores two MCK related aspects of the preservice teachers' classroom experience. First, the study investigates the MCK that secondary mathematics preservice teachers choose to enact (or choose not to enact) in practicum lessons. Enacted MCK in this study refers to observable manifestations of aspects of the preservice teachers' mathematical knowledge, such as written demonstrations of mathematical procedures or verbal references to particular mathematical concepts. Second, the elements of the decision making process that lead to MCK related teaching actions are studied. Elements of the decision making process comprise both the components of instructional decisions concerning MCK and the factors that lead preservice teachers to make those decisions. The examination of MCK related decisions and actions provide insights into the type and quality of MCK that preservice teachers choose to share with their students in a live lesson and how particular elements impact the mathematical content that they decide to deliver.

1.1 Background to the study

Concerns regarding the quality of mathematics teacher education both in Australia and abroad have been raised in recent years. Studies of mathematics teacher education programs around the world (Adler & Davis, 2006; Hsieh et al., 2011; Schmidt, Cogan, & Houang, 2011; Tatto, Lerman, & Novotna, 2010) have suggested that current programs may not be meeting the needs of preservice teachers. In Australia, calls for a change in teacher education programs echo those abroad and have come from a number of key stakeholders, such as the Australian Mathematics Trust and the Australian Academy of Technological Sciences and Engineering (Lawrance & Palmer, 2003), and more recently, Professor Ian Chubb, the Chief Scientist of Australia (Chubb, Findlay, Du, Burmester, & Kusa, 2012). One significant concern regarding current programs locally and globally is how effectively they equip future teachers with the professional knowledge needed to deliver high-quality instruction (Baumert et al., 2010; Lloyd, 2013). MCK is a fundamental type of professional knowledge that preservice teachers need to teach mathematics effectively (Ball, Hill, & Bass, 2005; Cooney, 1999) and that is why MCK is the focus of this study.

Attempts to determine the kind of mathematics knowledge needed by preservice teachers stem from Shulman's (1986; 1987) influential work on teacher knowledge. Shulman (1987) described seven categories of teacher knowledge: content knowledge, general pedagogical knowledge, curriculum knowledge, pedagogical content knowledge, knowledge of learners, knowledge of educational contexts, and knowledge of educational end, purposes, and values. Deep and multifaceted knowledge of discipline content was identified by Shulman (1986; 1987) as an important element of teacher knowledge and much of the research in the area of mathematics teacher preparation over the last three decades has consequently been drawn from his work. Building on the work of Shulman, Deborah Ball and her colleagues (Ball et al., 2005; Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008) examined more closely the discipline knowledge of mathematics teaching and suggest that the mathematical knowledge needed for effective teaching may be more specialised than first thought. Their assertion that mathematical knowledge be studied within the context of teaching has resulted in the development of the notion of mathematical content needed specifically for mathematics teaching (Adler, 2005; Hill, Ball, et al., 2008; Krauss et al., 2008).

The *Mathematical Knowledge for Teaching* (*MKfT*) framework developed by Ball and her colleagues (Ball & Bass, 2009; Ball et al., 2008) demonstrates the multidimensional nature of content knowledge that mathematics teachers should ideally hold. The framework reconceptualises and elaborates three of Shulman's (1987) seven categories of teacher knowledge: content knowledge, pedagogical content knowledge, and curricular knowledge, specifically for mathematics teaching. The framework comprises six subdomains clustered under two broader knowledge domains, namely pedagogical content knowledge (PCK) and mathematical content knowledge (MCK).

PCK refers to “amalgam knowledge that combines the knowing of content with the knowing of students and pedagogy” (Ball et al., 2008, p. 398). MCK is closely related to, but distinct from PCK and is described as “a specialised form of pure subject matter knowledge – ‘pure’ because it is not mixed with knowledge of students or pedagogy …and ‘specialised’ because it is not needed or used in settings other than mathematics teaching” (Ball et al., 2008, p. 396). Studies measuring teachers’ mathematical knowledge for teaching and their students’ achievements at the primary school level (Hill, Rowan, & Ball, 2005) and secondary school level (Baumert et al., 2010) have identified a significant positive relationship, highlighting the value of these important types of teacher knowledge.

The domain of PCK is a strong indicator of high quality mathematics instruction (Baumert et al., 2010; Hill, Ball et al., 2008). Measures of primary and secondary mathematics teachers’ PCK have been captured using responses to teaching scenario questions, sometimes accompanied by follow up interviews (An, Kulm, & Wu, 2004; Baumert et al., 2010; Beswick & Goos, 2012; Chick, Baker, Pham, & Cheng, 2006; Goos, 2013; Hill, Ball, et al., 2008). In some instances, PCK measures were then compared with student achievement gains (Baumert et al., 2010), homework and assessment tasks (Baumert et al., 2010), or observed classroom teaching episodes (Hill, Blunk et al., 2008). Promising positive correlations were found between PCK measures, effective teaching behaviours, and student achievement gains. PCK in these studies was measured from teacher responses to artificial scenarios, so it cannot be ascertained whether teachers would have responded to the same scenarios in the same way, had they been situated in a live classroom context. Despite PCK being measured outside the classroom, the results of the aforementioned studies indicate that the quality of a teacher’s PCK is likely to influence the quality of their mathematical instruction.

The strength of a teacher’s PCK relies, in part, on the knowledge they have of mathematics content. In a large study of almost two hundred secondary teachers, completed by German researchers as part of the COACTIV study to investigate teacher competence, Baumert et al. (2010) identified PCK as having “greater predictive power for student progress” (p. 164) and being a more influential factor for the quality of instruction than mathematics knowledge alone. However, the researchers cautioned that despite PCK being a more decisive factor in instructional quality, they regarded it as “inconceivable without a substantial level of content knowledge” (Baumert et al., 2010, p. 163). Baumert and his colleagues’ claim highlights the close association of MCK with PCK, and this view is echoed by other mathematics education scholars (Capraro, Capraro, Parker, Kulm, & Raulerson, 2005; Even, 1993; Goos, 2013; Harel, Fuller, & Rabin, 2008; Thomas, 2003). The level of mathematical knowledge held by a teacher is therefore a significant underlying factor impacting the quality of mathematics instruction and an essential knowledge type for preservice teachers to develop. It is for this reason that preservice teacher MCK is investigated in this study.

Australian preservice teachers appear to be failing to develop an adequate understanding of MCK during the course of their teacher education programs. Research concerning Australian teacher education programs suggests that preservice teachers may lack the mathematical resources needed to be effective teachers. Tatto et al. (2010) concluded after reviewing the teacher education programs of over 20 countries, including Australia, that there was a relatively low emphasis of mathematical content in Australian secondary teacher education programs. They suggested that the likelihood of graduating secondary teachers from programs such as these holding a low level of mathematical knowledge was a "possible troubling trend" (Tatto et al., 2010, p. 321). The concerns of Tatto et al. (2010) were reflected by Australian preservice teachers themselves in an earlier study by Kanes and Nisbet (1996) who found that fewer than half of the secondary mathematics preservice teachers surveyed believed that they possessed sufficient mathematics knowledge to teach successfully. More recently, Goos (2013) found in a study of 100 Australian secondary mathematics preservice teachers that their knowledge of secondary school content was "not necessarily secure" (p. 982).

Given that teachers' mathematical knowledge is closely associated with student achievement (Blömeke & Delaney, 2012; Linsell & Anakin, 2012), the need for the knowledge limitations of preservice teachers to be addressed in teacher education programs is therefore now evident and necessary (Cooney, 1999; Goos, 2013). To address those limitations adequately, however, the mathematical strengths and limitations of secondary mathematics preservice teachers need to be more fully understood, including how preservice teachers' MCK manifests in live teaching actions and how preservice teachers decide on the content they deliver.

1.2 Rationale for the study

Little research currently exists on measuring the MCK that secondary preservice teachers choose to enact in a live classroom context. Data used to describe and measure secondary preservice teachers' MCK have predominantly been generated in contexts outside the secondary classroom, in the form of responses to written surveys or interviews (Ball, 1990; Baumert et al., 2010; Bryan, 1999; Even, 1993; Goos, 2013; Stump, 1999). Responses to survey items or interview questions provide an indication of the MCK that preservice teachers may hold of one or more mathematics topics, but they do not adequately capture the situated nature of mathematical knowledge. Adler and Pillay (2007) argue that discussions of mathematical knowledge for teaching, including MCK, require consideration of "its specificity within a particular context of practice" (p. 88) due to the complex nature of the live classroom context.

One of the objectives of this study is to examine the MCK that manifests in secondary preservice teachers' live teaching actions within the practicum context. The practicum is a "fundamental aspect of most undergraduate teacher preparation programs" (Markworth, Goodwin, & Glisson, 2009, p. 67) and provided the opportunity for the researcher to study the MCK that preservice teachers call upon during instruction. The possibility of collecting data pertaining to live teaching actions presented a number of unique and valuable research opportunities. For example, within the practicum context, which is considered a high stakes venture by preservice teachers (Sim, 2011), would preservice teachers take additional steps to strengthen their own mathematical knowledge for particular topics that they teach before their lessons? How would they decide on the mathematical content they deliver at different points in their lessons? Are preservice teachers prepared to share all of the MCK they possess with their students? To more adequately describe preservice teachers' MCK for the work of teaching, MCK related decisions and teaching actions were investigated within the classroom context.

The study investigated preservice teachers' delivery of lower secondary algebraic content during practicum lessons. Students need to be successful in algebra, which has been referred to as the gate-keeper to gain access to senior secondary school mathematics and more advanced mathematics (Capraro & Joffrion, 2006; Kilpatrick, Swafford, & Findell, 2001; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Martinez, Brizuela, & Superfine, 2011; Silver, 2000). Ferrini-Mundy, Floden, McCrory, Burrill, and Sandow (2005) argue that teachers require particular mathematical knowledge for teaching algebra and empirical studies grounded in the practice of teaching are needed to understand how teachers use this knowledge.

This study contributes to the limited body of literature about preservice teachers MCK of algebra that is presented during instruction. A review of the literature yielded a limited number of studies that specifically measured aspects of secondary preservice teachers' MCK of algebra outside the classroom (Bryan, 1999; Even, 1993; Goos, 2013; Stump, 1999) and even fewer that investigated preservice teachers' application of their MCK of algebra in secondary algebra lessons (Markworth et al., 2009; Rowland, Jared, & Thwaites; Thwaites, Jared, & Rowland, 2011). The methodological design of this study allowed for the type and quality of MCK that preservice teachers deliver in live algebra lessons to be examined. Hence, the findings contribute valuable insights into prospective teachers' MCK of algebra. However, to study only the preservice teacher's MCK in the teaching act is not sufficient because the teaching act itself may influence the MCK that is visible. To capture the contextualised nature of MCK, this study also investigated the decisions that led to preservice teachers delivering particular MCK.

The MCK that preservice teachers enact is considered, in this study, to be the result of MCK related decisions made by the preservice teachers before and during their lessons. Shavelson and

Stern (1981) claim that focusing only on the actions of teachers in the classroom is "conceptually incomplete" (p. 455) and that consideration of the decisions teachers make in the classroom are needed to understand why teachers undertake particular teaching actions. To adequately describe the MCK evident in a live lesson, it was therefore necessary to examine the decisions that led to the MCK related teaching actions. Thus, this study investigated both the preservice teachers' decisions pertaining to MCK and the type and quality of MCK that the preservice teachers enacted as a result of those decisions.

1.3 Purpose of the study

The purpose of this study was to investigate the MCK of algebra that manifests in preservice teachers' live teaching actions and the thoughts behind those actions. To capture the visible aspects of the phenomenon, that is, the MCK that preservice teachers deliver, observation data were collected from six preservice teachers as they taught a total of ten lower secondary algebra lessons during their practicums. To capture the aspects of the phenomenon that were not visible in the lessons (i.e., the thoughts that led to actions), stimulated recall interviews with the preservice teachers provided the primary source of data. By examining both the enacted MCK of preservice teachers and the thoughts behind the enactment, it was possible to make judgements about the preservice teachers' choice of MCK for teaching algebra. Two research questions were developed to explore the MCK related thoughts and actions of preservice teachers:

1. What elements influence the decisions secondary preservice teachers make regarding the mathematical content knowledge (MCK) they enact when teaching lower secondary algebra?

2. What is the mathematical content knowledge (MCK) that secondary preservice teachers enact when teaching lower secondary algebra?

1.4 Significance of the study

The study makes three significant contributions to research about preservice teachers' MCK. First, the study contributes to knowledge about the decision making that leads to particular MCK being taught in the classroom. The study revealed the influencing elements of the MCK related decision making process that led preservice teachers to present higher quality and lower quality MCK of algebra in their lessons. It also explored the decision making elements that competed for the preservice teachers' attention and the influences that were prioritised or discarded as MCK related decisions were made. Because the study provides insights into the complexity of the MCK related decision process for secondary mathematics preservice teachers, it can inform change in the preservice teacher education experiences of future secondary mathematics teachers.

The second contribution made by this study is to the body of knowledge regarding the MCK that preservice teachers call upon in the context of the practicum classroom. The type of MCK that preservice teachers do and do not teach is discussed, including making qualitative judgements concerning the adequacy of their enacted MCK for teaching lower secondary algebra. Thus, a description of preservice teachers' MCK for teaching secondary algebra that is grounded in the live practicum classroom context is provided. The description of preservice teachers' MCK in context suggests the need for changes in preservice teacher education, especially concerning how preservice teachers' MCK might be better developed for teaching.

The third contribution this study makes is the methodological approach designed to capture the visible (i.e., teaching actions) and the invisible (i.e., thoughts behind teaching actions) aspects of preservice teachers' enacted MCK. Aspects of introspective (Boring, 1953; Hurlburt & Heavey, 2006) methodological approaches were employed to generate data pertaining to the preservice teachers' MCK related thoughts and actions. Taking an interpretivist (Erickson, 1986; Mason, 1996) approach to the analysis of the data, a detailed analytical framework, based on the literature concerning MCK and teacher decision making, was designed and applied. The methodological approach produces a "reasonable" measure of the MCK that manifests when preservice teachers deliver lower secondary algebra lessons and the influences that appear to impact preservice teachers' decisions to teach particular MCK of algebra during instruction.

1.5 Sequence of the study

This dissertation comprises seven chapters, with chapter 1 having outlined the rationale, purpose, and significance of the study. Chapter 2 presents an extended discussion of empirical and theoretical studies concerning two fields of research: MCK and teacher decision making. A review of the literature regarding MCK that mathematics teachers should hold and enact, and which was operationalised in this study, is first presented. Next, a review of studies that investigated preservice teacher MCK is discussed, highlighting a need for more contextualised studies of MCK and investigation of the elements that lead preservice teachers to share certain MCK with students in a live lesson. A review of teacher decision making frameworks that have been used to describe preservice teachers' decisions, and that were used to inform the methodological design of the study, follows the discussion of preservice teachers' MCK. Finally, the preservice teacher's ability to make high quality MCK related decisions is explored with respect to the literature concerning preservice teachers' prior educational experiences.

Chapter 3 presents the study's theoretical perspective that forefronted enacted MCK and which led to the study's methodological approach. The study design is described, including data collection processes and techniques, data analysis phases, and analysis frameworks. Ethical considerations and the limitations of the methodological process used complete the chapter.

Chapters 4 and 5 present the findings of the study. Categories of influencing elements impacting preservice teachers' MCK related decisions are presented first in chapter 4. The MCK that preservice teachers subsequently enacted as a result of those decisions and the influencing elements that were associated with each type and quality of enacted MCK are reported in chapter 5, the second of the two findings chapters.

Chapter 6 synthesises the findings of the previous two chapters and discusses the findings in light of the literature presented in chapter 2. Chapter 7 concludes the dissertation by recapping its findings and presenting implications for practice as well as directions for further research.

Chapter 2: Review of the literature

2.0 Introduction

This study investigated the mathematical content knowledge (MCK) that preservice teachers enact in lower secondary algebra lessons. The study also explored the preservice teachers' thoughts that lay behind their MCK related teaching actions. Examining both the MCK related thoughts and actions together in a single research project positioned the study at the intersection of two broad fields of research: MCK and teacher decision making. A review of the literature pertaining to each of these research fields is presented in this chapter, which is organised in four major sections.

The first two major sections of this chapter focus on MCK related teaching actions. The first section of the chapter reviews the different types and domains of MCK that pertain to teaching lower secondary algebra. It describes the "ideal" MCK that scholars argue should feature in secondary mathematics lessons and that the study operationalised for the analysis of the preservice teachers' MCK related teaching actions. The second section critically reviews studies that describe secondary mathematics preservice teachers' MCK. The review in this section reveals the limitations associated with investigating MCK outside the classroom, establishing the context in which this study is located and the need to investigate the thoughts behind preservice teachers' MCK related teaching actions.

The third and fourth major sections of this chapter focus on the literature pertaining to preservice teachers' MCK related decisions, the thoughts most closely associated with MCK related teaching actions. The third section presents a review of the literature on novice teacher decision making. Frameworks of teacher decision making that have been used to investigate the decisions of novice (graduate) or pre-novice (preservice) teachers are discussed, with an emphasis on influencing elements that either form part of, or contribute to, teachers' instructional decisions. The fourth and final section of this chapter extends the discussion of preservice teachers' decision making provided in section 3. This section reviews the literature concerning preservice teachers' mathematics and mathematics teaching experiences and how they might impact the quality of the MCK related decisions they make.

2.1 The MCK of teachers

The extended discussion in this major section reviews the scholarship in the field of MCK. The section begins by providing a rationale for using the literature concerning expert teachers' MCK to analyse and describe preservice teachers' MCK in this study. The next two sections discuss three types of MCK that relate specifically to the teaching of lower secondary algebra:

conceptual knowledge, procedural knowledge, and algebraic ways of thinking (AWOTS). The final section reviews the literature on the unique domains of mathematical knowledge held by mathematics teachers, outlined by Ball and her colleagues in their *Mathematical Knowledge for Teaching (MKfT)* framework (e.g., Ball & Bass, 2009; Ball et al., 2008; Hill, Blunk et al., 2008).

This study required a theoretical model of MCK by which to gauge the MCK that preservice teachers enact in practicum lessons. A review of the literature concerning MCK yielded no detailed theoretical models of preservice teacher MCK. The *Knowledge Quartet* framework, developed by Rowland, Huckstep, and Thwaites (2005) to study preservice teachers' mathematical teaching actions does refer to subject matter knowledge and according to the researchers, the framework has "a focus on the teacher's mathematical content knowledge" (Thwaites et al., 2011). However, as the authors themselves noted, they were more interested in using the framework to classify situations in which MCK surfaced, than to describe the kinds of mathematical knowledge enacted by a preservice teacher (Rowland, 2008). Thus, no details regarding the types of subject matter knowledge or the nature of that knowledge specifically for preservice teachers are included in the *Knowledge Quartet* framework or were found in a search of mathematics education literature. To inform the design of an analytical framework which could be used to describe preservice teachers' enacted MCK, the researcher drew from the literature relating to expert teachers' MCK.

A review of the scholarship in the field of MCK revealed a number of theoretical constructs employed to characterise "ideal" MCK. These constructs describe the mathematical knowledge potentially held by an experienced and effective secondary mathematics teacher and were used to analyse the "visible" aspects of the preservice teachers' MCK related classroom actions. The use of expert MCK to describe preservice teachers' MCK had its limitations because the participants had little chance of enacting MCK that was equivalent to that of an experienced teacher. This limitation is confirmed by novice to expert models of professional performance such as that of Dreyfus and Dreyfus (1980) and Berliner (2001). These scholars indicate that novices do not hold their knowledge or put their knowledge into practice in the same way as experts. Therefore, the expert MCK described in this section was included in the framework with the intention of providing a benchmark against which the preservice teachers' enacted MCK might be compared.

MCK has been decomposed in a number of ways by various researchers, with each version highlighting different key aspects of what constitutes quality MCK for teaching. Scholars have identified specific facets of mathematics content that mathematics teachers should draw upon in their teaching (e.g., Driscoll, 1999; Harel, 2008c; Hiebert & LeFevre, 1986; Skemp, 1976), whilst others have focused on the different types of specialisation that teachers hold with respect to their mathematics knowledge (e.g., Ball et al., 2008; Ma, 1999; Star & Stylianides, 2013; Wu, 2008). De Jong and Ferguson-Hessler (1996) recommend separating characteristics relating to the type of knowledge and the quality of knowledge when considering knowledge-in-action. Following this advice, this study drew from both perspectives offered in the literature.

The description of MCK synthesised from the literature and used in this study presents the mathematical knowledge needed to teach lower secondary algebra from two, closely related, perspectives. First, the type of mathematical content that teachers should ideally hold will be outlined using three major categories identified from the literature: conceptual knowledge, procedural knowledge, and knowledge of AWOTS. Secondly, the way in which effective teachers should hold those three knowledge types will be described, using the sub-domains of the *MKfT* framework (Ball et al., 2008). The descriptions of MCK that follow do not capture all the possible types of MCK for all secondary mathematical topics, or even all of those MCK types related to lower secondary algebra. For example, problem solving approaches that are suitable for use in a lesson on problem solving in an algebraic context and geometric ways of thinking that would be beneficial to use for a graphical approach to solving algebraic equations were beyond the scope of this study. What the MCK description does capture is the significant types of MCK that would typically appear in lower secondary algebra lessons and the ideal form that knowledge should take in teacher practice.

2.1.1 Procedural and conceptual knowledge

The mathematics that teachers enact and that students learn within the schooling context is generally conceived of in two categories: conceptual knowledge and procedural knowledge, with overlap between the two categories considered highly advantageous (Lindmeier, 2011). In the mathematical proficiencies of the Australian curriculum, based on earlier work by Kilpatrick et al. (2001), conceptual and procedural knowledge are highlighted directly in the proficiencies of *understanding* and *fluency*, respectively (Australian Curriculum Assessment and Reporting Authority [ACARA], 2015). In addition, these two knowledge types are required for success in the remaining two proficiency strands of the Australian curriculum, *problem solving* and *reasoning*, where students develop proficiencies such as applying an existing procedure to an unfamiliar situation or justifying a conclusion that has been reached (ACARA, 2015).

The popularity of these terms stem from Hiebert and Le Fevre’s (1986) work. More recent analyses have extended what is considered by some as an underdeveloped view of these two types of knowledge by Hiebert and Le Fevre (Baroody, Feil, & Johnson, 2007; Star & Stylianides, 2013). Both conceptual and procedural knowledge are acknowledged as vital, despite debates existing for more than a century over which kind of knowledge is perceived as more valuable and worthy of emphasis in instructional programs (Hiebert & Carpenter, 1992; Star, 2005). What follows is a description first of procedural knowledge and then of conceptual knowledge that teachers could reasonably be expected to manifest in their teaching practice of lower secondary algebra lessons.

2.1.1.1 Procedural knowledge

Procedural knowledge is one type of MCK used to examine preservice teachers' MCK in this study. The term was popularised by Hiebert and Le Fevre (1986), but similar phrases such as "knowing-how" by Ryle (1949/2000, p. 28) and "knowings-how" by Skemp (1979, p. 170) precede the more well-known term by several years. Hiebert and Le Fevre's original characterisation of procedural knowledge includes two distinct kinds of knowledge. The first kind comprises "the formal language, or symbol representation system, of mathematics" (Hiebert & Le Fevre, 1986, p. 6) while the second consists of "rules, algorithms, or procedures used to solve mathematical tasks" (p. 7). Both kinds of knowledge are considered elements of procedural knowledge for this study.

In the context of a lower secondary algebra lesson, preservice teachers might enact their knowledge of a number of algebraic procedures related to a given topic. Friedlander and Arcavi (2012) describe five procedures for beginner algebra students, namely, manipulating algebraic expressions, solving equations and inequations, solving systems of equations, factorisation, and operations with negative numbers. All procedures identified by Friedlander and Arcavi (2012) were present in one or more of this study's participants' lessons, but the first three procedures typified the majority of algebraic procedures taught by the preservice teachers in this study. With respect to solving equations, three procedural methods used to solve simple linear equations were pertinent to this study.

One method used to solve linear equations is the backtracking (Green, 2009; Pearson Australia, 2011) or undoing (Kieran, 1992) method. This method requires operations performed on a pronumeral to be identified and sometimes recorded one by one in a flowchart (Pearson Australia, 2011), spreadsheet (Green, 2009), or other graphical displays such as onion rings (Passy's world of mathematics, 2012). Inverse operations are then performed in reverse order on the number equivalent to the expression involving the pronumeral, to find the solution (see Appendix A, method 1 for an example of the method). A second method is the balance method (Linchevski & Hercovics, 1996; Vlassis, 2002; Wu, 2010) where the same operation is applied to the expressions on both sides of an equation, ensuring that the (equivalent) equations produced in each new line of working remain "balanced" each time (refer to Appendix A, method 2 for an example). This method produces a series of transformed equations that are equivalent to each other (Steinberg, Sleeman, & Ktorza, 1991). A third method is the transposing method (Wu, 2010) which involves transposing a term from one side of an equation to another using inverse operations (Appendix A, method 3 shows an example of this method). Hall (2002) refers to this method as "a change side – change sign technique" (p. 12). Procedures and sub-procedures pertaining to solving equations using these three methods (including representing a situation with an equation first) and manipulating simple algebraic expressions forms the basis of the mathematical context in which this study took place.

2.1.1.2 Conceptual knowledge

Conceptual knowledge in this study is considered a necessary and crucial type of teacher MCK. De Jong and Ferguson-Hessler (1996, p. 107) define conceptual knowledge as "knowledge about facts, concepts, and principles that apply within a certain domain." Examples of these for teachers include concept definitions (Usiskin, 2001; Wu, 2008), concept representations (Davis, 2008a; Kilpatrick et al., 2001; Vlassis, 2004), essential features and general principles underpinning concepts (Even, 1993; Skemp, 1976), and any associated mental pictures or analogies (Davis, 2008a; Tall & Vinner, 1981). Together, these examples contribute to the formation of what Tall and Vinner (1981) refer to as "concept images" (p. 151), which are individuals' personal cognitive structures of mathematics concepts. Not surprisingly, a robust, comprehensive, and connected set of concept images is considered by all authors as advantageous for algebra teachers to possess. The mathematical concepts most closely related to the topics presented by the participants in their lessons in this study relate to the dual nature of algebraic objects, the varying roles of the pronumeral, and understanding of operations.

Mathematical concepts can be interpreted *operationally* and *structurally*, according to Sfard (1991), and algebra students need to develop both perspectives. The interpretation of algebraic expressions, equations, and the equals symbol as both a process (operationally) and an object (structurally) is critical in algebra. Studies of students' understanding of the equals symbol have found that students who are unable to view the equals symbol from a structural perspective, as a sign of equivalence, find limited success in algebra (Knuth, Stephens, McNeil, & Alibali, 2006; Norton & Cooper, 2001; Norton & Irvin, 2007; Prediger, 2010). The treatment of mathematical concepts from both perspectives is an important consideration for teachers so the study data were analysed with this consideration in mind.

The role of the pronumeral in algebra varies according to the mathematical context in which it is situated (Wu, 2010). The variety of meanings for the pronumeral, x, include a dependent or independent variable, a generalised number, a specific unknown number, the solution, a set of numbers, a parameter, or a constant (Ely & Adams, 2012; Horne, 2005; Kieran, 1992;

Küchemann, 1978; Wu, 2010). Teachers must take care to refer to pronumerals according to the context in which they are mediated in different lessons, to reduce student confusion and to ensure that students do not adopt only one meaning for pronumerals (Horne, 2005) or mistake letters in algebra for object labels such as "a" for apple (Christou, Vosniadou, & Vamvakoussi, 2007). Care with reference to pronumerals was taken into account when the preservice teachers' instructional actions in this study were analysed.

Competence in algebra requires a substantial understanding of the general properties of number, including basic operations (Norton & Cooper, 2001). Successful manipulation of expressions and solving of equations requires knowledge of numbers and operations that include the following:

- associative and commutative laws of addition and multiplication and the distributive law of multiplication over addition and subtraction (Kieran, 1992; Welder, 2006; Wu, 2010);
- inverse operations (Linchevski & Herscovics, 1996; Pjanić & Nesimović, 2013);
- the principle of cancelling (Falle, 2005);
- conventions regarding the order of operations (Kieran, 1992; Norton & Cooper, 2001; Welder, 2006);
- fraction concepts (Norton & Irving, 2007);
- operations with integers and directed numbers (Bofferding, 2010; Gallardo, 2002; Herscovics & Linchevski, 1994; Norton & Irving, 2007; Vlassis, 2002).

Each concept listed above would already have been taught to the students in the study's preservice teachers' classes. Because they relate directly to students' success in algebra, explicit reference to these concepts was included in the analysis of the preservice teachers' MCK related actions.

Teachers need to teach procedural and conceptual knowledge in algebra lessons. The breadth, depth, and connectedness of these two types of knowledge are discussed later in this chapter section. It is not only the teachers' knowledge of concepts and procedures which deserve attention in algebra lessons. The cognitive characteristics underpinning them offer a unique and valuable lens through which to study preservice teachers' MCK related actions.

2.1.2 Knowledge of algebraic ways of thinking

The cognitive characteristics associated with mathematical concepts and procedures are referred to as mathematical "ways of thinking" (Harel, 2008b, p. 894). At a very inexperienced level of mathematical thought, ways of thinking such as additive thinking (Young-Loveridge & Mills, 2011) or multiplicative thinking (Siemon, 2005) exist. As a person is exposed to new and more advanced mathematical topics, concepts, and procedures, new ways of thinking about those topics, concepts, and procedures also develop. While some general ways of thinking refer to multiple content strands, such as "mathematicians talk small and think big" (Cuoco, Goldenberg & Mark, 1996, p. 384), for the purposes of this study, ways of thinking will be limited to those that directly relate to an algebraic way of thinking about mathematics.

Algebraic ways of thinking (AWOTS) refer to certain mathematical habits that come to mind when thinking algebraically (Driscoll, 1999). Thinking algebraically can include using symbols, solving equations, or performing algebraic manipulations (Dindyal, 2007; Driscoll, 1999). These were the topics taught by the study's preservice teachers in their lessons. When individuals are exposed to a certain way of thinking over time, they will develop a tendency to approach and make sense of mathematical situations with that way of thinking. The disposition to act according to certain tendencies is well captured by Lim and Seldon's (2009) description of ways of thinking as "habituated" (p. 1578) and is related to the more popularised term, "habits of mind," emphasised by Cuoco, Goldenberk, Mark, and Hirsch (2010, p. 682). AWOTS are an important inclusion in the MCK framework of this study because they allow the mathematical tendencies of preservice teachers to be investigated.

A review of related literature yielded five AWOTS that would be desirable for teachers to explicitly teach when attending to algebra topics. The broadest of all AWOTS is the *manipulating with purpose* way of thinking. Harel (2008b; 2008c) contends that algebraic symbols are manipulated with purpose, to arrive at a particular form. Explicitly articulating a mathematical goal for a procedure is particularly important when teaching mathematics, so that students gain a greater sense of the reason for performing certain procedures or sub-procedures. Silver (1997) argues that students need exposure to the "the important ideas that lie behind the seemingly endless list of procedures" (p. 206). One of those ideas is sharing the purpose behind algebraic manipulations.

The second algebraic way of thinking, closely associated to the first, is known as the *algebraic invariance* way of thinking. This way of thinking, according to Harel (2008c), is a tendency to purposefully manipulate an algebraic expression by changing the form of the expression but holding another property of the expression (such as the value of the expression) constant or invariant. Cuoco et al. (2010) refer to this way of thinking as "purposefully transforming…expressions" (p. 686) which they argue should be pointed out to students to reveal the utility of algebraic manipulations.

The third algebraic way of thinking, espoused by Driscoll (1999), is called the *doing-undoing* way of thinking. Driscoll and Moyer (2001) explain that effective algebraic thinking involves not only understanding a process to achieve a mathematical goal (i.e., *manipulating with purpose*) but also understanding a process so well that it can be reversed. The backtracking method of solving equations (Appendix A, method 1), for example, would require this way of thinking. Closely related notions in the literature are referred to as "reverse thinking" (Friedlander & Arcavi, 2012, p. 610) and "working backwards" (Kieran, 1992, p. 393; Pjanić & Nesimović, 2013, p. 216).

The fourth algebraic way of thinking, *building rules to represent functions* (Driscoll, 1999, p. 2), is referred to in several scholars' work (Capraro & Joffrion, 2006; Day & Jones, 1997; Moses, 2000). Driscoll's (1999) description of this way of thinking refers to recognising patterns in mathematical situations and representing those situations with an input-output functional rule. An operational view of algebraic equations is emphasised in Driscoll's (1999) description. In this study, the *building rules to represent functions* way of thinking is broadened to encompass the capacity to represent worded scenarios involving input-output situations and other types of mathematical relationships, such as those in Capraro and Joffrion's study (2006), symbolically.

The final algebraic way of thinking captures algebraic thinking at its most abstract. The *abstracting from computation* (Driscoll, 1999, p.2) way of thinking, known by Mark, Cuoco, Goldenberg, and Sword (2010) as "abstracting regularity from calculations" (p. 506), refers to "think[ing] about computations freed from the particular numbers they are tied to in arithmetic" (Driscoll, 1999, p. 2). Developing this way of thinking helps students learn to generalise patterns and express them in symbolic form (Pjanić & Nesimović, 2013), transitioning from concrete thinking to more abstract ways of thinking.

Cuoco et al. (2010) argue that ways of thinking should be explicitly taught to students, justifying their inclusion in the MCK analysis framework of this study, alongside procedural and conceptual knowledge. Together, knowledge of concepts, procedures, and ways of thinking form the three types of preservice teacher MCK that were investigated. Each type of MCK is limited in this study to the aspects which relate most directly to the topics presented by the participants in their lower secondary algebra lessons. The next section explores the sub-domains of MCK provided by Ball and her colleagues (e.g., Ball et al., 2008; Ball & Bass, 2009), to investigate how procedures, concepts, and ways of thinking can be known first, by non-teachers and teachers alike, and second, known in special ways that are unique to mathematics teachers.

2.1.3 Domains of MCK held by effective mathematics teachers

MCK needed specifically for mathematics teaching has gained the attention of many researchers in recent years (Even, 1990; Hill, Ball, et al., 2008; Kajander et al., 2010; Krauss et al., 2008; Zazkis & Leikin, 2010). A variety of terms that refer to MCK include "subject matter knowledge" (Ball et al., 2008, p. 389), "mathematics-for-teaching" (Davis, 2008a, p. 5), and "teachers' mathematics" (Usiskin, 2001, p. 87). Although the language differs, there is strong agreement amongst these scholars that, as Davis (2008b) argues, teachers not only need to know more mathematics than their students but to know it differently. Galbraith (2008) and Usiskin (2001) go further to argue that teachers' mathematics is in fact a unique branch of applied mathematics, "where the field of application is education" (Galbraith, 2008, p. 5). Whilst many authors acknowledge the uniqueness of the mathematical knowledge needed by teachers, Ball and her colleagues suggest that MCK needed for effective teaching is insufficiently understood (Ball & Bass, 2000; Thames, Sleep, Bass, & Ball, 2008) inviting further exploration of the substance and nature of MCK and how it shapes teachers' actions. This study contributes to the body of literature concerning MCK by using what is known about MCK to analyse preservice teachers' instructional actions.

For each type of MCK investigated in this study, it is possible to hold that knowledge in the same way as a non-teacher might know mathematics or alternatively, in specialised ways needed for teaching. The three subdomains of MCK, according to the *MKfT* framework (Ball et al., 2005; Ball et al., 2008; Ball & Bass, 2009), introduced in chapter 1, capture this distinction and are used to organise the discussion in the remainder of this chapter section. Common content knowledge (Ball et al., 2005) of algebraic procedures, related concepts, and AWOTS, i.e., knowledge that a non-teacher might possess, is first explored. The chapter section then presents literature pertaining to specialised forms of procedural knowledge, conceptual knowledge, and ways of thinking that are needed specifically for teaching algebra. Teacher-specific forms of MCK are presented in terms of the two MCK subdomains, specialised content knowledge (Ball et al., 2005) and knowledge at the mathematical horizon (Ball & Bass, 2009).

2.1.3.1 Common content knowledge

Common content knowledge provides a foundation for more specialised forms of mathematical knowledge needed for teaching to develop. It has been described as mathematical knowledge that is commonly required in many mathematically demanding professions, including, but not exclusive to teaching (Ball et al., 2008; Hill, 2010). Within a mathematics classroom context, teachers' common content knowledge is the mathematical knowledge needed to complete the age-appropriate work that they would assign their students (Ball et al., 2008; Hill, 2010). Examples include accurate recall of mathematical terms, notation, procedures, and concept definitions (Ball et al., 2008; Rowland, Turner, Thwaites, & Huckstep, 2009). Knowledge of this type has immediate application in mathematics classrooms because teachers are regularly called upon to identify terms, notate ideas with mathematical symbols, or model procedures.

The two subsequent sub-domains of MCK, namely, specialised content knowledge and knowledge at the mathematical horizon, are extensions of the common content knowledge held by mathematics teachers. These two sub-domains rely, in part, upon a robust knowledge of the mathematics content that would be known broadly across professions. However, they also depend on teachers' ability to know more mathematics than non-teachers in certain situations and to adapt their mathematical knowledge to better suit the work of teaching. Additions and modifications to a substantial common content knowledge base are needed for teachers to cope effectively with the demands of teaching school mathematics.

2.1.3.2 Specialised content knowledge

Mathematical knowledge that goes beyond the common content knowledge expected of a well-educated adult is referred to by Ball et al. (2008) as specialised content knowledge. This dimension reflects the unique way that teachers need to understand mathematics. For example, precision regarding the presentation of concepts, procedures, and AWOTS are needed by teachers in particular because precise mathematical language is used by effective mathematics teachers (Evertson, Emmer, & Brophy, 1980; Good & Grouws, 1977; Lim, 2007). What makes this mathematical knowledge so specialised requires both a quantitative and qualitative description because teachers not only need to know more mathematics, but they need to hold that knowledge in a different, more accessible form. Ball and her colleagues have not explicitly summarised the elements and nature of specialised content knowledge. Instead, they reference certain knowledge types and qualities when summarising the tasks of teaching which involve this knowledge dimension (Ball et al., 2008). With specific reference to algebra, McCrory, Floden, Ferrini-Mundy, Reckase, and Senk (2012), in their framework of knowledge needed for teaching secondary algebra, also describe typical practices that algebra teachers undertake that rely heavily on MCK and they note their similarity to elements of specialised content knowledge described by Ball et al. (2008). It was through the tasks that were described to illustrate these two frameworks and further supplemented by additional literature (e.g., Ma, 1999; Rowland et al., 2009) that the specialised aspects of MCK for teaching secondary school algebra, used to analyse the participants' actions in this study, were gleaned.

Specialised content knowledge, in this study, possesses four interrelated features that differentiate it from common content knowledge. Teachers with specialised content knowledge know more mathematics content at a *broader* and *deeper* level and they know it in a more *connected* and *decompressed* way than a person with only common content knowledge. The features, which are each explicated below, were identified repeatedly in a review of the literature on the mathematical knowledge that teachers should ideally hold when teaching mathematics, and more specifically, when teaching secondary algebra. The first two features, knowing content more broadly and deeply, captures a quantitative increase in the knowledge held by a teacher compared with a non-teacher, whereas the remaining two features, relating to connecting and decompressing mathematical knowledge for teaching, are qualitative descriptions of specialised content knowledge. The features of specialised content knowledge together indicate a dynamic, ongoing process of knowledge refinement, rather than a static end product.

Expert teachers' specialised content knowledge, from a purely quantitative perspective, comprises a larger knowledge base than a novice teacher or non-teacher would possess. In layman's terms, expert teachers simply know more about the content they are teaching. Teachers are encouraged to develop depth and breadth of mathematical understanding and both qualities relate to the relative conceptual strength of the additional knowledge. It is valuable for teachers to hold additional mathematical knowledge of similar, more, and less cognitive levels to common content knowledge. According to Ma (1999), this allows for mathematics to be known both more broadly and more deeply. Ma attributes breadth and depth of mathematical knowledge to effective teachers, suggesting that additional knowledge can positively impact the work that teachers do.

Broad MCK

Broad mathematical content refers to the extra mathematical knowledge that is no more conceptually powerful than the content to which it is related (Ma, 1999). A teacher relies upon a breadth of knowledge when they call upon alternative definitions, representations, procedures, or ways of thinking that they may find useful to enact when teaching in a particular circumstance. Rowland et al. (2009) explain that this is why specialised content knowledge is "wider" (p. 153) than common content knowledge. A number of teaching tasks which Ball et al. (2008) argue require specialised content knowledge involve choosing from a range of options. For example, a teacher often chooses a particular representation of a concept (e.g., a vinculum or obelus to represent division) or a solution path to solve an equation that they judge is the most appropriate for his or her class in a particular moment. A necessary prerequisite for this kind of decision then, is for the teacher to have, at their disposal, knowledge of multiple concept representations or multiple strategies for approaching a problem. This may be superfluous knowledge for a non-teacher, who has no need to weigh up the strengths and weaknesses of multiple options as teachers often are required to do. This knowledge is different from and in addition to common content knowledge. Therefore, a characteristic of specialised content knowledge of teachers is breadth of knowledge.

Mathematical knowledge known broadly for teaching can refer to additional knowledge of the concepts, procedures, or mathematical ways of thinking. Ma (1999) and Tall and Vinner (1981) state that multiple forms of knowledge about concepts, including formal definitions, mental images and associated properties, which Tall and Vinner (1981) refer to as “concept images” (p. 152), are highly beneficial for teachers to possess. Additional knowledge of procedures, including alternative solution paths, are valuable additions to teacher knowledge, according to Ball et al. (2008), who describe one of the tasks of teachers as judging the relative usefulness of different solution paths. Multiple ways of thinking algebraically are implicit in effective teachers’ actions described by Driscoll (1999). He contends that teachers should choose questions carefully, with the intention of eliciting particular ways of thinking in student responses. A knowledge of the variety of AWOTS that exist are needed for Driscoll’s advice to be put into practice successfully. The addition of alternate conceptual knowledge, procedural knowledge, or AWOTS can each contribute to a broader type of mathematical knowledge, offering a teacher more flexibility in terms of the knowledge that they have at their disposal during instruction.

Deep MCK

Depth of understanding, according to Ma (1999), occurs when knowledge is connected with “more conceptually powerful ideas” (p. 121). Ball et al. (2008) describe this knowledge as the concepts that underpin mathematical facts, representations, language, and procedures. They, and others, argue that knowledge of concepts or sub-concepts that support a procedure or example allow teachers to make judgements about a student’s non-traditional solution method or to explain why a certain procedure or operation is useful in a particular situation (Ball et al., 2008; Morris, Hiebert, & Spitzer, 2009). It is not breadth of knowledge in this case which is called upon but depth of knowledge. When teachers understand more deeply the topics they teach in the classroom, they are able to cope more successfully with the mathematical demands placed on them during instruction. It is essential for teachers, therefore, to increase the depth and breadth of their knowledge base, by taking steps to augment the mathematical content they already possess prior to teaching, with similar and more conceptually profound ideas. Complementing the quantitative aspects of specialised content knowledge are two qualitative descriptors: connected MCK and decompressed MCK, which are each now described.

Connected MCK

Connectedness is a qualitative characteristic of specialised content knowledge that allows teachers to make the most of the additional knowledge they have in their practice. It is widely regarded as a key feature of deep mathematical understanding for teachers and non-teachers alike. Mathematics education researchers repeatedly equate connected mathematics knowledge with a strong understanding of the discipline, arguing for mathematics to be understood as a cohesive network, rather than a collection of isolated concepts, procedures, and topics (Hiebert & Carpenter, 1992; Kilpatrick et al., 2001; Ma, 1999; Mhlolo, Venkat, & Schäfer, 2012; Skemp, 1979; Star, 2005). Chinnappan, Lawson, and Nason (1999) describe the benefits of having a connected way of knowing mathematics, explaining that "the quality of connections among knowledge components is assumed to influence the ease with which the presence of one element aids in the retrieval and use of another in a problem environment" (pp. 167-168). Given that teaching has been conceptualised as a special form of mathematical problem solving (Mason & Spence, 1999), connections that promote both knowledge retrieval and adaptable mathematics thinking are resources that teachers can draw upon in their practice.

The benefits associated with teachers relying upon numerous and rich connections to inform their practice is emphasised regularly in the literature (Ball & Bass, 2009; Ball et al., 2008; Chinnappan et al., 1999; Gagne, 1985; Ma, 1999; Mason & Spence, 1999), current Australian teaching standards (AITSL, 2014), and Australian curriculum documents (ACARA, 2015). The Australian Professional Standards for Teachers require teachers to "organise content into an effective learning and teaching sequence" (AITSL, 2014, standard 2.2). Connected knowledge would be an asset to any teacher required to perform this task, including graduates, for whom the quote specifically refers. Teachers who hold connected mathematical knowledge would also be well placed to meet the demands of the Australian curriculum for mathematics because mathematical connections feature within the aims, content structure, and secondary mathematics achievement standards (ACARA, 2015). At the lesson planning level, connections allow teachers to recognise previously learned concepts and procedures that would support new learning. For example, Tall (1989; 1992) introduces the notion of a cognitive root, which he describes as an "anchoring concept" (Tall, 1989, p. 40) that teachers present to enhance the introduction of new mathematics concepts. In the moment of teaching, connected knowledge enables teachers to better exploit any opportunities to review important concepts or prepare students for later ones (Ma, 1999). Hence, connected mathematical knowledge is an important qualitative aspect of specialised content knowledge that has the potential to improve the practice of teachers.

Connected knowledge can vary according to the elements of mathematics knowledge that are connected and the richness of those connections. Potential exists for multiple knowledges to be known in connection with each other. For example, knowledge of a concept may be connected to other related concepts (ACARA, 2015; Kilpatrick et al., 2001; McCrory et al., 2012), procedures (Gray & Tall, 1994; Kilpatrick et al., 2001; Skemp, 1979), or ways of thinking (Harel, 2008c). In an algebraic context, the concept of equivalence might be connected to 'equation' (related concept), the balance method to solve an equation (related procedure), and the advantage of creating equivalent equations (the algebraic invariance way of thinking). It is also possible to know the steps involved in the balance method independently of the concept of equivalence, an understanding of equivalent equations, or an alternative method of solving an equation. However, for teachers, isolated mathematical knowledge eliminates even the possibility that they can make connections explicit to their students. Ideally, related aspects of teachers' MCK should be connected, creating the potential for them to call upon those connections in their classroom practice. Not all connections, however, are perceived in the literature as of equal importance.

Despite the many similarities with Hiebert and Le Fevre's (1986) conceptualisation of procedural knowledge, the stance taken of procedural knowledge in this study differs in one key respect, that is, the depth at which procedural knowledge can be held. Perceptions of procedural knowledge known independently of conceptual knowledge are rarely favourable but arguments exist for the relative utility of possessing distinct, flexible procedural knowledge. Rarely foregrounded in mathematics education research (Star, 2007), procedural knowledge is regarded by some as the poor cousin of conceptual knowledge. Restrictive views of procedural knowledge exist, such as the parallels identified with rote knowledge and memorization (Eisenhart et al., 1993; Pesek & Kirshner, 2000; Skemp, 1979) and the perception that it is devoid of the rich relationships which characterise conceptual knowledge (Hiebert & Le Fevre, 1986). Although the overwhelming opinion of mathematics education researchers is that connections between conceptual and procedural knowledge are essential (Eisenhart et al., 1993; Gray & Tall, 1994; Hiebert & Le Fevre, 1986; Kilpatrick et al., 2001; Schneider, Rittle-Johnson, & Star, 2011; Skemp, 1979), research undertaken by Star and his colleagues in the last decade (Schneider et al., 2011; Star, 2005; Star, 2007; Star & Seifert, 2006; Star & Stylianides, 2013; Yakes & Star, 2011) highlights the value of procedural knowledge in its own right.

Procedural knowledge, in this study, is understood to be more than memorisation and regurgitation. Ideally, it should be known deeply in a connected way. Ryle (1949/2000) argues against the categorisation of procedural knowledge as a simplistic form of knowledge, using the analogy of well-trained circus seals, who are able to reproduce tricks flawlessly but are unable to regulate their own actions. A person of intelligence, according to Ryle (1949/2000), could additionally "detect and correct lapses [and]…repeat and improve upon successes" (p. 29). Ryle indicates that strong procedural knowledge involves more than only reproductions of routine procedures and symbols. Star and Seifert (2006) found strong procedural knowledge existed in their study of Year 8 students' solution methods to linear equations. The students' flexible and nuanced approach to equation solving revealed additional procedural knowledge beyond that required to reproduce an algorithm and included knowledge of multiple strategies and of their respective strengths and weaknesses. Star and Seifert's (2006) findings echoed Greeno's (1978) theoretical analysis of procedural knowledge, where he contended that "to know a procedure, a person must know an action and know the condition in which the action should be performed" (Greeno, 1978, p. 271). Star (2005; 2007) subsequently reconceptualised Hiebert and Le Fevre's (1986) definition of procedural knowledge, which he considered was too superficial and devoid of the rich connections possible, and argued that procedural knowledge be considered as potentially deep, with multiple connections providing procedural flexibility.

It is Star's (2005) notion of potentially flexible and connected procedural knowledge which is used in this study. Procedural expertise, identified by Ball et al. (2008) as beneficial for effective teachers to possess, would require a well-developed and connected knowledge of procedures, rules, symbols, and language as described by Star, rather than the relatively superficial view of procedural knowledge, as outlined by Hiebert and Le Fevre (1986). The methodological design of this study allowed for the possibility of any specialised procedural knowledge enacted by preservice teachers to be identified, independently of the possible added presence of conceptual knowledge or ways of thinking.

Connections between two or more aspects of procedural knowledge are valuable additions to teachers' MCK but would be more beneficial if conceptual knowledge was also involved. Connected procedural knowledge is generally regarded positively in the literature, underpinning a flexible approach to procedures (Baroody et al., 2007; Hiebert and Le Fevre (1986); Kilpatrick et al., 2001; Skemp, 1979; Star, 2005, 2007). Despite the recognized benefits of possessing connected procedural knowledge, and in particular Star's (2007) argument that deep procedural knowledge can be known independently of conceptual knowledge, scholars agree that substantial connections with conceptual knowledge are even more valuable and lead to a deeper understanding of why procedures work as they do (Kilpatrick et al., 2001; Lederman & Niess, 1997; Rowland et al., 2009; Skemp, 1979). Skemp (1979) refers to this kind of connected procedural and conceptual knowledge as "relational understanding" (p. 259), while Ma (1999) uses the term "knowledge package" (p. 115). Conceptual connections can also allow for further connectedness between different ways of thinking and procedures, such as connecting algebraic and geometric perspectives when optimising the area of a rectangle (de Araujo et al., 2013). Connections involving conceptual knowledge are consequently regarded as richer and more favourable than those involving procedural knowledge alone.

Connected conceptual knowledge is widely regarded as essential for mathematics teaching, even when considered apart from knowledge of procedures or ways of thinking. Hiebert and Le Fevre’s (1986) description of conceptual knowledge refers not only to the knowledge of facts, concepts, and principles, but also to the rich connectedness of that knowledge. The strong emphasis on the connected nature of conceptual knowledge is well supported in mathematics education literature (Chinnappan et al., 1999; Even, 1990; Greeno, 1978; Kilpatrick et al., 2001; Ma, 1999; Vinner, 1983). For example, in the work of teaching, knowing how different representations are similar and different, and choosing appropriate ones for particular teaching episodes, relies upon the teacher knowing the conceptual connections inherent in the representations (Ball et al., 2008; Kilpatrick et al., 2001). Thus, connected conceptual knowledge is one of the central qualitative aspects of specialised content knowledge.

Decompressed MCK

One of the unique aspects of specialised content knowledge is that teachers hold their mathematical knowledge in a decompressed state. This is described by Ball et al. (2008) as understanding in a “self-conscious way” (p. 400). Ma (1999) and Skemp (1979) use the analogy of a taxi driver, who knows a town differently to a newcomer or a local, who stick to the routes they know. In contrast, the taxi driver draws upon his or her decompressed knowledge of the town, which includes the features of multiple routes, and is able to choose from a range of different routes depending on a desire for scenery or speed. Skemp (1979) indicates that decompressed knowledge increases “adaptability” (p. 170) because knowing the features of each path can help in designing a successful plan for a particular journey. Similarly, when mathematical knowledge is decompressed, underlying features among procedures, concepts, topics, and the contexts in which they are located are more easily identifiable. These unpacked features assist teachers in making and carrying out successful lesson plans in mathematics.

Expert mathematics teachers hold their knowledge in a decompressed state and use it to make expedient decisions regarding the mathematics they enact. In a similar fashion to taxi drivers, expert mathematics teachers can make choices about which mathematical journey they wish to offer their students, choosing representations, tasks, examples, and solution paths that would be most beneficial for their class (Adler & Davis, 2006; Ball et al., 2008; Chick, 2009). Their awareness of the features and mathematical principles inherent in different representations, examples, or solution paths allow them to make mindful decisions about enacting or omitting certain mathematical concepts, procedures, or ways of thinking. If teachers are able to unpack mathematical examples, procedures, and concepts, the mathematical ideas of each can be made more easily apparent to students (Ma, 1999; Kilpatrick et al., 2001). Expert teachers can potentially offer well directed explanations or student feedback by explicitly and consciously highlighting particular mathematical ideas. Skemp's (1976) notion of "simplicity by unifying" (p. 25), which he claims can be achieved by "penetrating beyond superficial differences" (p. 25), reflects the potential value of actions involving decompressed content knowledge.

Holding algebra in a decompressed form is a worthy goal for secondary mathematics teachers. McCrory et al. (2012) state that "decompressing" (p. 601) is one of three major categories of secondary algebra teachers' work. Teaching students how to perform an algebraic procedure is only part of the work that algebra teachers should be undertaking, according to McCrory et al. (2012), because they need to teach explicitly the pertinent ways of thinking and mathematical concepts inherent in the procedures. They argue that algebra teachers should "decompress procedures students or possibly they themselves have learned by rote, helping students grasp the logic of the procedures (and recognize important restrictions and limiting cases)" (McCrory et al., 2012, p. 602). Students need their teachers to hold their MCK in a decompressed form if they are to be exposed to the mathematical principles behind procedures. Students also need their teachers to know mathematics in a form that is suitable to share with inexperienced mathematics students, yet is still sensitive to its location within the wider discipline of mathematics, highlighting the third and final dimension of MCK for teachers.

2.1.3.3 Knowledge at the mathematical horizon

The third and most recent dimension of MCK identified in the *MKfT* framework (Ball et al., 2008) is knowledge at the mathematical horizon or as it is referred to in this study, horizon knowledge. Ball and Bass (2009) describe horizon knowledge as an awareness of the "mathematical landscape in which the present experience and instruction is situated" (p. 6), indicating that attention is paid by teachers not only to the content they are specifically engaging with in a lesson but to related ideas that lie beyond it. Teachers with horizon knowledge consider lesson content from a broader viewpoint than a particular topic for a particular grade. They possess what Ball and Bass (2009) call a kind of "peripheral vision" (p. 1), noting features of concepts or procedures that are connected to several contexts, either across mathematical strands or real life contexts (de Araujo et al., 2013). Possessing this knowledge helps teachers to anticipate mathematical connections or distortions, notice or evaluate mathematical opportunities, and prepare their students effectively for more advanced mathematical ideas (Ball & Bass, 2009). In doing so, they are more able to customise content to suit their students in a way that demonstrates mathematical integrity.

Horizon knowledge allows teachers to customise elements of mathematics for individual lessons or classes but without losing sight of the discipline of mathematics as a whole. While it is preferable to view mathematics as an expansive and interconnected network of ideas, teachers must isolate and reduce mathematical content for pragmatic reasons. With time and curriculum restrictions in mind, teachers take a slice of the discipline of mathematics to share with their students in any lesson. That isolated slice can realistically include only a limited number of concepts, procedures, or ways of thinking. Teachers must then further adapt the content as they know it, to present it in ways that students can intellectually manage. This process is described in secondary algebra classes as "trimming" (McCrory et al., 2012, p. 604).

Trimming involves "scaling down or up, intentionally omitting or adding detail, or modifying levels of rigor. Trimming also involves recognizing mathematics that has been trimmed too much, namely, instances in which important details or special cases are missing" (McCrory et al., 2012, p. 604). Customising content in these ways are daily teaching tasks that should be undertaken in what Bruner (1977) refers to as an "intellectually honest" (p. 33) manner, where reduced content does not distort how mathematics might be viewed beyond the mathematical content of the lesson. Teachers therefore must keep an eye on the mathematical horizon to effectively trim and not over trim mathematical content for their lessons.

Knowledge that maintains the integrity of the discipline of mathematics has been identified by several mathematics education researchers as being part of teacher knowledge (Ball & Bass, 2009; Clemens, 1991; Harel, 2008a; Lim, 2008; Ma, 1999; Schifter, 2001; Wu, 2006). They are unanimous in their assertions that it is necessary to customise mathematical knowledge for a school setting but that it must be done "without sacrificing mathematical integrity" (Wu, 2006, p. 8). Attempting to meet both conditions creates a mathematical balancing act that teachers must perform as they interact with their students. If either mathematical integrity or the intellectual needs of the students are neglected, poor teaching results (Harel, 2008a). Failing to preserve the mathematical integrity of the content presented by enacting an imprecise or contextually limited version of mathematics would distort how mathematics is viewed by students and would undoubtedly impair student learning. Ball and Bass (2003) provide the converse argument, arguing that a mathematical definition "is useless, no matter how mathematically refined or elegant, if it includes terms that are beyond the prospective user's knowledge" (p. 8). The challenge facing teachers then, is to know and enact mathematical knowledge in a form that meets students' intellectual needs, but does not distort mathematical meaning.

A review of literature related to Ball and Bass' (2009) notion of horizon knowledge revealed it has three qualitative characteristics. Firstly, effective mathematics teachers have a connected understanding of mathematics beyond that of specialised content knowledge. Secondly, teachers with well-developed horizon knowledge have additional knowledge of mathematics that is sensitive to the limitations of the classroom context in which the mathematics they are presenting is situated. Underpinning both of these characteristics is a third characteristic, namely the decompression of mathematical knowledge. Horizon knowledge should be held in a decompressed form in much the same way as teachers hold specialised content knowledge, so further explanation of this characteristic is not provided. The additional complexity of teachers' connected knowledge and contextually sensitive knowledge are now further explored.

The connected nature of horizon knowledge bears similarities and differences to the connected nature of specialised content knowledge. Both horizon knowledge and specialised content knowledge indicate how crucial it is for teachers to be aware of the connections between different mathematical representations, examples, or procedures by recognising the mathematical features inherent in each one. Horizon knowledge requires an understanding of how these connections are located within the larger discipline of mathematics; where they might first be encountered and where they might lead to in later mathematics or be applied in different contexts (Ball & Bass, 2009). In the context of algebra teaching, McCrory et al. (2012) use the term, "bridging" (p. 606), to describe horizontal and vertical connections in the curriculum. Horizon knowledge includes connections across mathematical contexts and topics (i.e., horizontal connections), back to less sophisticated mathematical ideas, and forwards to more complex ideas (i.e., vertical connections). In this way, horizon knowledge extends the sense of connectedness that exists in specialised content knowledge to beyond the content of the lesson.

Contextually sensitive content, developed specifically for mathematics learners, is a characteristic of horizon knowledge. The expectation to communicate mathematics on a lower conceptual level (Skemp, 1971) requires teachers to reconceptualise the mathematics knowledge which they hold; to know mathematics differently from non-teachers. When teachers reconceptualise their own MCK to better suit their work as teachers, they develop alternative versions of mathematical knowledge that are cognitively simpler and therefore more accessible to their students. Examples include alternative definitions, procedures, or representations that are mathematically less demanding than their more advanced counterparts. When a teacher has horizon knowledge, the alternative versions maintain mathematical integrity by aligning with and not distorting the content and nature of the discipline of mathematics.

Open mathematical ideas that are contextually sensitive are held by teachers with horizon knowledge. Open or broad mathematical definitions are important for mathematics teachers to hold (Davis, 2008a, 2008b; Usiskin, 2001; Wu, 2006). Open definitions take into account the development of a concept over time and with increasing levels of mathematical sophistication in the school curriculum (Davis, 2008a). They comprise different interpretations that are needed for different stages of mathematical development and allow for revision and extension as new situations are encountered (Davis, 2008b). Ball et al. (2008) argue that teachers should develop alternative but equivalent definitions so that "useable definitions" (p. 400) can be chosen judiciously and with respect to the broader discipline of mathematics. Open definitions are consistent with Ball and Bass' (2009) notion of horizon knowledge for teachers and with Ma's (1999) description of thorough mathematics knowledge which creates "the capability to pass through all parts of the field" (p. 121). By developing an awareness of the contextual restraints that limit the application of closed definitions, teachers can work towards using open definitions, allowing concepts to unfold over time with a higher degree of mathematical integrity maintained.

In summary, common content knowledge, specialised content knowledge, and horizon knowledge pertaining to procedural knowledge, conceptual knowledge, and mathematical ways of thinking constitute the mathematical knowledge that expert teachers use in their practice. Common content knowledge provides a sound foundation of MCK for teaching. Specialised content knowledge and horizon knowledge reflect an increased knowledge of mathematics in a form that is decompressed and accessible, connected in multiple ways, and contextually sensitive to the broader discipline of mathematics. These dimensions describe the ideal type and nature of mathematics knowledge for teaching which preservice teachers should develop. This study seeks to investigate which aspects of their MCK might be developing at this stage of their teaching career when the mathematics in their teaching actions is explored.

2.2 Enacted MCK of preservice teachers: Research to date

The process of putting one's knowledge into practice in a particular context is referred to in this study as enacting knowledge. When knowledge is enacted, people act on their knowledge in such a way as to give it concrete shape (Orlikowski, 2002; Rooney, 2005). In the context of the classroom, MCK that takes concrete form manifests in verbal utterances, body movements, or written communications. Observations of enacted knowledge cannot be construed to represent all that a person knows for a particular topic because contextual circumstances contribute to the selection of knowledge a person decides to enact (Schoenfeld, 2010). Observations of enacted knowledge offer an indication of the knowledge that a person may have for a given topic and what they decided to put into practice when placed in a particular situation or context.

Preservice teachers enact MCK inside and outside classrooms. Within the classroom context, enacting knowledge refers to the actions undertaken by a teacher which put that teacher's knowledge into practice. These actions take the form of behaviours such as verbal explanations, questions and prompts, or written notes, questions, or feedback. Examples of preservice teachers enacting MCK during instruction include the provision of tasks, explanations, questions, or demonstrations that specifically relate to one or more aspects of mathematics content. Outside the context of the classroom, preservice teachers may enact MCK in any number of contexts but they would do so without consideration of a live classroom experience in real time. One of the premises of this study is that the live classroom experience influences preservice teachers' MCK related decisions. A distinction therefore is maintained in the review of studies provided in this chapter section between measures of MCK taken inside and outside the classroom context.

A literature search was undertaken to identify empirical studies that described the MCK enacted by preservice teachers. An initial search for studies that focused specifically on both the live classroom experience and secondary preservice teachers' MCK yielded only five articles (Borko & Livingston, 1989; Livingston & Borko, 1990; Markworth et al., 2009; Rowland et al., 2011; Thwaites et al., 2011). The literature search was subsequently expanded to include studies of (a) only secondary preservice teachers' MCK and (b) primary and secondary preservice teachers' MCK (within a single study), from contexts outside the live classroom. After an initial search, a systematic search was undertaken of the following journals from the years 2000 to 2015:

- American Educational Research Journal;
- The Australian Journal of Teacher Education;
- Canadian Journal of Science, Mathematics, and Technology Education;
- Educational Studies in Mathematics;
- For the Learning of Mathematics;
- Journal for Research in Mathematics Education;
- Journal of Mathematics Teacher Education;
- Journal of Teacher Education;
- Mathematics Education Research Journal;
- Mathematics Teacher Education and Development Journal;
- Research in Mathematics Education;
- ZDM: The International Journal on Mathematics Education.

The literature search yielded 34 studies in total. Five of those studies examined MCK taught in practicum lessons and the remaining 29 reported findings based on data collected outside the classroom context. The methodological approaches and findings of the two sets of studies are described in the following two sections.

2.2.1 Descriptions of preservice teachers' MCK outside the classroom

The majority of studies investigating secondary preservice teacher MCK described manifestations of secondary preservice teacher MCK outside the live classroom. For the 29 studies situated outside the practicum classroom setting, the instruments used to gauge preservice teachers' MCK comprised the following:

- Written or online mathematics test only (11 studies);
- Written/online test and university-based activities (e.g., responses to questions, lesson plans, lesson reflections, analysis of hypothetical student responses, or teaching resources) (6 studies);
- Written/online test and follow up interviews (5 studies);
- University-based activities (e.g., responses to questions, lesson plans, lesson reflections, analysis of hypothetical student responses, or teaching resources) (4 studies);
- Interview only (3 studies).

A written or online mathematics test was the data collection instrument used in almost all studies (22 out of 29 studies), allowing for the collection of data from large cohorts of preservice teachers across multiple tertiary institutions (e.g., Ball, 1990; Goos, 2013; Huang, 2014). The number of participants in the studies ranged from only one secondary preservice teacher (Plummer & Peterson, 2009) to over five thousand (Tatto et al., 2012). Across all studies, regardless of sample size or choice of data collection instruments, the researchers set the mathematical direction of the MCK that preservice teachers enacted by their choice of test questions, interview questions, or types of mathematical tasks they gave to the participants. The choice by the researchers in these studies to investigate particular aspects of preservice teacher MCK differs from the approach taken in this study, where the researcher chose to examine whatever MCK emerged in the participants' teaching actions during their lessons.

The 29 "out of classroom" studies found in a general search of the literature and a detailed search within the 12 journals identified previously, assessed MCK that involved knowledge of lower secondary algebra either in its own right or as part of other strands of secondary mathematics. Nine studies focused only on lower secondary algebra content, 11 studies included algebra as part of multiple topics being investigated, three studies included the participants' use of algebra in the solution of questions from other mathematical strands, and six studies focused on more advanced topics such as functions or calculus. A common conclusion across the studies, including those that focused only on algebra, was that the participants lacked conceptual understanding of mathematics topics (e.g., Bryan, 1999; Even, 1993). Preservice teachers also appeared to have significant difficulties when they attempted to solve problems involving relationships between multiple concepts (Goos, 2013; Tatto et al., 2012). Where MCK of algebra was specifically concerned, preservice teachers exhibited a strong preference for procedural approaches, (Plummer & Peterson, 2009; Ticknor, 2012), mechanical treatment of algebraic procedures (du Toit, 2009; Latterell, 2008), and limited conceptual knowledge (Ball, 1990; Stump, 1999). Overall, the findings of these studies indicate that preservice teachers may not possess a deep or connected knowledge of many mathematical topics that they will be teaching to their students, including lower secondary algebra.

A limitation of the MCK descriptions in these studies, noted by some of the authors themselves, is that they are based on data that are removed from instructional practice (Kahan, Cooper, & Bethea, 2003). Although studies of this kind can hone in on particular MCK for examination, which is a strength of developing testing instruments with particular mathematical foci, it is not possible to say from the findings whether the same mathematical strengths or lapses might be present in preservice teachers' practicum lessons. For example, it seems likely that a preservice teacher who could not solve a particular question type would take steps to rectify his or her mathematical inadequacies if required to model a solution to that question to a class of students. Less certain is whether preservice teachers are aware of, or attend to, any conceptual deficiencies when they prepare for their lessons. It is also not possible to say from these findings whether preservice teachers would enact the same MCK in a university test as in a practicum classroom, when faced with real students. As Chapman (2013) argues, investigations are needed into how MCK impacts classroom actions because it is within those classroom actions that MCK for teaching can be most clearly seen (Rowland et al., 2009). Hence, to judge preservice teachers' MCK for the work of teaching, studies such as this one, which investigates the MCK that manifests in preservice teachers' actions, are needed to supplement the findings of the "out of classroom" studies of secondary mathematics preservice teacher MCK.

2.2.2 Descriptions of preservice teachers' MCK inside the classroom

Five articles featuring descriptions of secondary preservice teacher MCK in a live classroom were found in the literature. The MCK descriptions were provided as part of findings related to preservice teachers' mathematical and pedagogical knowledge and in only one study was preservice teacher MCK described in terms of common content knowledge, specialised content knowledge, or horizon knowledge (Markworth et al., 2009). The articles were based on case studies of one (Markworth et al., 2009; Rowland et al., 2011; Thwaites et al., 2011) or two (Borko & Livingston, 1989; Livingston & Borko, 1990) secondary preservice teachers as they taught up to five mathematics lessons during a single practicum phase. In two of the studies (Borko & Livingston, 1989; Livingston & Borko, 1990), the preservice teachers' instructional actions were compared with those of their supervising teachers. The content delivered in the lessons consisted of direct proportion, calculus, analytical geometry, and algebra (including solving simultaneous linear equations, solving and sketching quadratic equations, and factorising polynomials). The number of lessons reported in the articles varied between a single lesson (Rowland et al., 2011; Thwaites et al., 2011), two lessons (Livingston & Borko, 1990), five weekly lessons over consecutive weeks (Markworth et al., 2009), and one week of consecutive lessons (Borko & Livingston, 1989).

The data collected by the researchers in these studies comprised lesson observations and interview data. Live lessons were observed in the five studies and lesson data were recorded using the researchers' field notes either on their own (Markworth et al., 2009) or in addition to audio recordings (Borko & Livingston, 1989; Livingston & Borko, 1990) or video recordings (Rowland et al., 2011; Thwaites et al., 2011). Interviews were conducted with all preservice teachers in the studies but varied according to the type of interview and timing of the interviews. In the study conducted by Markworth et al. (2009), which included a weekly lesson observation for five weeks, interviews with one of the researchers only took place at the start and finish of the practicum phase. Following the observed lessons, rather than interviews, audiotaped conversations between the participant and her supervising teacher about the lesson were conducted. Post-lesson interviews were undertaken in the remaining four studies. The studies by Rowland et al. (2011) and Thwaites et al. (2011) utilised stimulated recall procedures. The post-lesson interviews usually took place soon after the lesson was completed, although in one study, the interview occurred 20 days after the lesson (Rowland et al., 2011). In two studies (Borko & Livingston, 1989; Livingston & Borko, 1990), pre-lesson interview data were also collected. Lesson artefacts, such as lesson plans and related class textbook content, completed the data collected in these studies.

The data analyses employed in the studies of secondary preservice teacher MCK during instruction included references to MCK but MCK was not the sole focus of the research questions or the subsequent data analyses. In the studies by Borko and Livingston (Borko & Livingston, 1989; Livingston & Borko, 1990), the nature of novice and expert mathematics teaching expertise was investigated, including the MCK that preservice teachers (the novices of the study) enact. Categories and subcategories of preservice teachers' live mathematics teaching actions (e.g., statements or examples) and their post interview thoughts were developed during the data analysis phase. The analyses included references to both the type of MCK that the preservice teachers enacted and the relative quality of MCK they enacted in different class circumstances. Other teacher knowledge types were also analysed in these studies.

The remaining three studies used theoretical frameworks based on previous empirical studies to analyse the preservice teachers' actions when teaching mathematics. Markworth et al. (2009) used three sub-domains of the *MKfT* framework by Ball and her colleagues (Ball et al., 2008): specialised content knowledge, knowledge of content and students, and knowledge of content and teaching. The presence of two pedagogical content knowledge (PCK) sub-domains and the absence of common content knowledge and horizon knowledge indicate MCK was not the primary focus of the analysis. Rowland and his colleagues (Rowland et al., 2011; Thwaites et al., 2011) used the *Knowledge Quartet* framework to analyse the teaching actions of their participants. The framework was developed from their studies of primary mathematics preservice teachers' live teaching actions (Rowland et al., 2009) and is comprised of four knowledge dimensions: foundation, transformation, connection, and contingency. Content knowledge is located within the Foundation knowledge dimension of the framework. Once more, preservice teacher MCK was not the sole focus of the lesson analyses. Hence, the findings of the studies do include specific references to secondary preservice teacher MCK but do not systematically report the MCK enacted throughout a lesson or a fragment of a lesson.

The findings reported by each set of researchers reveal different aspects about preservice teacher MCK in practice. In the Borko and Livingston studies (Borko & Livingston, 1989; Livingston & Borko, 1990), references to preservice teachers' enacted MCK and preservice teachers' thoughts were discussed. The MCK enacted by the two secondary preservice teachers when they taught lessons on calculus and analytic geometry was described in general terms as mostly "accurate,… largely procedural, and…not conceptually linked" (Livingston & Borko, 1990, p. 384). The researchers also noted that mathematical explanations that were planned prior to the lesson tended to be higher in quality than those produced in the moment, in response to student questions. Unplanned explanations were described as mathematically incorrect and/or inadequate. At times, the participants admitted to avoiding a response to some student questions altogether because they were experiencing so many difficulties in responding with adequate explanations. They also held back from responding to students to ensure they completed their planned presentations. The findings of these studies suggest that MCK enacted in a live classroom is shaped, in part, by the circumstances in which preservice teachers find themselves and in the main, the circumstances produce poor MCK.

The two studies using the *Knowledge Quartet* framework to analyse lesson data (Rowland et al., 2011; Thwaites et al., 2011) revealed strengths and weaknesses in the MCK enacted. In the study by Thwaites et al. (2011), only brief references were made to enacted MCK but they revealed the preservice teacher's emphasis on why quadratic equations took particular graphical forms, with reference to earlier work in the lesson on completing the square. Enacted knowledge of why graphs took particular forms and not only on how to graph quadratic equations was a positive addition to the lesson, according to Thwaites et al. (2011). In contrast, the study findings of Rowland et al. (2011) identified enacted MCK of algebra that was less than ideal for teaching. Careless and incorrect use of mathematical terminology (e.g., "timesing") was noted in the preservice teachers' instructional actions alongside more precise and carefully chosen mathematical language (e.g., "coefficient"). The preservice teacher also reduced her explanation of the elimination method in solving equations simultaneously (whether scaling of the equations was needed or not) to a simplistic rule, "If the signs are the same, then subtract; if they are different, then add" (Rowland et al., 2011, p. 6). In her lesson reflection, the participant commented that the rule she had given appeared to be limiting the range of solution methods her students used. Overall, the studies by Rowland and his colleagues indicate that preservice teachers enact both positive and negative aspects of MCK during instruction.

The Markworth et al. (2009) study investigated changes in preservice teacher knowledge, including MCK, over a five week phase. Development of specialised content knowledge over the practicum phase was a notable MCK related finding of this study. The researchers found that over the five weeks, the secondary preservice teacher in the study learnt additional ways to perform procedures, thereby developing a broader knowledge of mathematical content, an aspect of specialised content knowledge. The researchers also noted that the preservice teacher developed a more refined understanding of mathematical terminology needed for teaching (e.g., the use of the terms monomial, binomial, trinomial, and polynomial).

In summary, few studies have investigated secondary preservice teacher MCK in the live classroom context. The reports of secondary preservice teacher MCK within the classroom context reveal significant issues regarding enacted MCK but are limited in three ways. Firstly, the reporting of preservice teacher MCK from only one or two participants reduces the generalisability of the findings of these five studies. Secondly, the delivery of content from different mathematical strands means that commonalities and differences in specific mathematical content that preservice teachers tended to enact when teaching certain topics could not be reported. Thirdly, although the studies did make reference to the MCK delivered during instruction, a systematic analysis of the enacted MCK was not included because the focus of the studies included other teacher knowledge types. As seen in the section 2.2.1, more detailed reports of secondary preservice teacher MCK are provided in studies located outside the live classroom but in these studies the situated nature of mathematical knowledge enacted in a live classroom within live student interactions cannot be adequately captured. This study aimed to systematically investigate the MCK that secondary preservice teachers enact during live lessons. The study also sought to identify trends across preservice teachers and across lessons by focusing on lower secondary algebra lessons taught by six participants.

The invisible cognitive processes underlying more visible expressions of MCK-in-action require careful consideration, as teachers' thinking necessarily influences their actions in the classroom. Shavelson and Stern (1981) posit that "to understand teaching, we must understand how thoughts get carried into actions" (p. 457), indicating the need to consider the thoughts that lead to enacted MCK in the live classroom. Decisions have been identified as the thoughts leading directly to teaching actions (Schoenfeld, 2010; Simon, 1995; Sullivan, Clarke, Clarke, & Roche, 2013; Westerman, 1991). To understand, as fully as possible, how preservice teachers come to enact certain MCK, the literature relating to teacher decision making and particularly that of the preservice teacher is now presented.

2.3 Preservice teacher decision making

To access the less visible aspects of MCK related teaching actions, this study drew on the literature related to teacher decision making. The literature is reviewed here to identify potential influencing elements on preservice teachers' MCK related decisions. To reiterate, influencing elements in this study refer to the components of preservice teachers' instructional decisions concerning MCK and the factors that lead them to make those decisions.

A decision of any kind implies choice. Without choice, Leigh (1983) argues, there is no genuine decision making taking place. The two major components of MCK related decision making in this study are (a) the preservice teachers' choice of goal(s) and (b) their choice of an MCK related means by which to achieve their goal(s). These two components are identified in Schoenfeld's (2010; 2011) framework of human decision making, which has been used to analyse the teaching actions of expert and preservice secondary mathematics teachers (Schoenfeld, 2010).

The decision making process is central to the cognitive work of teachers because it relates directly to their actions in the classroom. Studies of teachers' cognition have revealed that the decision making process undertaken by teachers repeatedly in their daily work directly results in teaching actions being performed or considered but discarded (Leinhardt & Greeno, 1986; Schoenfeld, 2010; Simon, 1995). Those decisions are made before lessons begin (John, 2006; Schoenfeld, 2010; Sullivan et al., 2013; Westerman, 1991) and during live teaching sequences within a lesson (Shavelson & Stern, 1981; Simon, 1995; Zodik & Zaslavsky, 2008). Cobb, Yackel, and Wood (1991) posit that teachers think consciously about the teaching actions they are going to take when they make decisions. Studying their conscious thoughts about the teaching act can therefore help to explain why teachers do what they do. Hence, investigating the decision making process in this study provided a means through which to study the thoughts leading to enacted MCK.

Six teacher decision making frameworks that relate to mathematics teaching and preservice teachers (John, 2006; Leinhardt & Greeno, 1986; Schoenfeld, 2010; Shavelson & Stern, 1981; Simon, 1995; Westerman, 1991) were identified in the literature and are reviewed in this section. A conceptual framework, according to Miles and Huberman (1994), includes "factors, constructs, or variables – and the presumed relationships among them" (p. 18) and help to explain, either in graphical form or in narrative form, the phenomenon being investigated. For the six frameworks described in this section, which comprise graphical and narrative forms, the process, components, and contributing factors of the phenomenon of teacher decision making are presented. No conceptual frameworks were found in the literature that specifically described secondary mathematics preservice teachers' decision making so the literature search was broadened to include decision making frameworks concerning primary and secondary preservice teachers of any subject area and expert teachers of any subject area *if* the decision making framework had subsequently been used to analyse novice (i.e., graduate teacher) or pre-novice (i.e., preservice teacher) teaching actions. The six frameworks reviewed comprise three pertaining to preservice teacher decision making for primary mathematics (Leinhardt & Greeno, 1986; Westerman, 1991) or for any subject area (John, 2006) and three concerning expert teacher decision making (Schoenfeld, 1999; Shavelson & Stern, 1981; Simon, 1995) that have successfully been applied in studies of preservice teachers (Byra & Sherman, 1993; Schoenfeld, 2010) or novice teachers in their first year of teaching (Amador & Lamberg, 2013).

A critical examination of each framework was undertaken to identify potential influencing elements for secondary mathematics preservice teachers' use of MCK in their classroom practice. The examination revealed similarities and differences in the way that the researchers describe the decision making process and the influencing elements that impact teaching decisions. Conclusions drawn from the analysis of these frameworks informed the development of the decision making framework that was used to analyse the data in this study.

2.3.1 Decision making frameworks for novice and pre-novice teachers

The six conceptual frameworks of teacher decision making are reviewed in this section in three ways. Firstly, each framework is introduced with respect the model(s) of decision making that form part of the frameworks, the methodological approaches used to generate the models, and the central idea that is featured in each framework. Secondly the decision making process, including the components of teachers' decisions, that are identified in each framework are compared and contrasted. Thirdly, the factors identified in each framework as contributing to teacher decisions are compared and contrasted.

When teachers make live decisions during a lesson, those decisions are of two major types: preactive and interactive decisions. The language used to describe these two decision types comes from Westerman's (1991) framework, which features three types of teacher decisions originally named by Jackson (1968). Only two of those decision types, preactive and interactive decisions, lead directly to live teaching actions.

Preactive decisions, in this study, are those made prior to the lesson, in the planning or preactive stage, and are reinforced once the lesson begins, resulting in particular teaching actions. Interactive decisions are those made during the live lesson, known as the interactive phase of teaching, that were not planned before the lesson. Both preactive and interactive decisions are two types of in-the-moment decisions because a teacher either decides in the moment to go through with pre-planned decisions once the lesson is underway or decides instead to make a spontaneous, interactive decision. Decisions made after the lesson ends are referred to by Westerman (1991) as postactive decisions and concern decisions for future lessons that are made after having reflected on a completed lesson. Postactive decisions were not captured in the data set of this study and did not feature in any of the frameworks besides that of Westerman (1991). Therefore, the review of the frameworks will refer only to descriptions provided for preactive and interactive decisions.

Included within each framework are models of one or more aspects of teacher decision making. The models take either a diagrammatic form or comprise an explicit sequence of written steps and are supplemented with the theorists' written elaborations. The models within the six frameworks focus on either preactive, interactive, or both types of teacher decision making. An overview of the six decision making frameworks is presented in Table 1, ordered according to the type(s) of decisions, preactive and/or interactive (refer to the far right column), that feature in the models within each framework.

Within Table 1, the names of the models of teacher decision making that form part of the conceptual frameworks are first provided. Next, details of the methodological design used by the theorists are presented. Reviews of empirical studies that were synthesised to produce a framework are contrasted with empirical studies that featured single or cross-case causal network analyses (Miles & Huberman, 1994).

Table 1. Methodological approaches used to produce teacher decision making models

Author (date)	Model(s)	Participants if applicable	Data	Method
John (2006)	Model of the planning process for teachers	n/a	Empirical studies concerning expert and novice teachers' instructional planning	Synthesis of empirical studies produced a causal model of factors contributing to *preactive* decisions
Schoenfeld (1998; 1999; 2010; 2011)	Human, in-the-moment decision making model (Schoenfeld, 2010; 2011) Developed from Teaching-in-context model (Schoenfeld, 1998; 1999)	A set of three studies involving: 3 expert teachers (Aguirre & Speer, 1999; Schoenfeld, Minstrell, & van Zee, 1999) 1 secondary mathematics preservice teacher (Zimmerlin & Nelson, 1999)	Lesson observations (videotaped footage, lesson artefacts, field notes) Post-lesson interviews	No explanation given of how the original causal model of decision making (Teaching-in-context) was generated Teaching-in-context model used to analyse lessons and produce a second, refined causal model of the process of *interactive* decision making
Simon (1995)	Mathematics teaching cycle decision making model	One mathematics teacher educator (author) 26 primary preservice teachers	Lesson data for a whole class, constructivist teaching experiment (videotaped footage, colleague's field notes) Teacher educator's reflective notes Preservice teacher journals	Single-case causal networking produced a causal model of the process of *preactive* and *interactive* decision making

Table 1. (continued)

Author (date)	**Model(s)**	**Participants if applicable**	**Data**	**Method**
Shavelson and Stern (1981)	Model 1: Factors contributing to teachers' pedagogical judgements and decisions Model 2: Elements of teachers' planning of instructional tasks Model 3: Teachers' decision making during interactive teaching	n/a	Empirical studies concerning teachers' judgements, preactive decisions, and interactive decisions	Synthesis of empirical studies produced: Model 1: Causal model of factors contributing to teachers' judgements and decisions Model 2: Non-causal model of factors contributing to *preactive* decisions Model 3: Decision tree modelling of process of *interactive* decision making
Leinhardt and Greeno (1986)	Model 1: Planning nets Model 2: Action segments	8 expert primary teachers 4 primary preservice teachers	Multiple lessons observed over 3½ months (videotaped footage and field notes) Pre-lesson interviews Post-lesson interviews (some with stimulated recall)	Cross-case causal networking produced: Model 1: Decision tree modelling of process of *preactive* decision making Model 2: Decision tree modelling of process of *interactive* decision making
Westerman (1991)	Model of expert teachers' preactive and interactive decision making Model of preservice teachers' preactive and interactive decision making	5 primary expert teachers 5 primary preservice teachers	Pre-lesson interviews Lesson data for two lessons per teacher (videotaped footage, lesson artefacts, field notes) Post-lesson stimulated recall interviews	Cross-case causal networking, using grounded theory, produced causal models of factors contributing to *preactive* and *interactive* decisions

Causal network analyses involve extracting and interpreting variables, "streams of variables" (Miles & Huberman, 1994, p. 228), and interrelationships that lead to a particular outcome. Causal network analysis, in this study, refers to the analysis of lesson and interview data to produce a causal network of elements that either contribute to, or form part of, teaching decisions. The type of models generated as part of the frameworks are also included in the table. Certain models depict the *process* by which teachers make decisions using decision making trees (Miles & Huberman, 1994) or causal networks (Miles & Huberman, 1994), whereas other models summarise the *factors* that contribute to teacher decisions using causal or non-causal networks (Miles & Huberman, 1994).

Table 1 shows that only one framework model (John, 2006) focuses primarily on preservice teachers' preactive decisions. Schoenfeld's (2010) framework includes a model that focuses on interactive decision making but he only refers to preactive decisions in the supplementary text of the framework that accompanies the model. The remaining four frameworks include detailed descriptions of both preactive and interactive decision making. Each framework is now briefly elaborated according to the fundamental idea(s) emphasised by the researchers in their decision making frameworks.

Factors considered by preservice teachers when they make preactive decisions comprise the decision making framework developed by John (2006). This framework refers only to decisions made in the lesson planning or preactive phase. The graphical model that features within the framework identifies 29 factors which practising teachers consider concurrently when planning a lesson. The methodological design of this study did not limit the potential contributing factors of preservice teachers' decisions to only these 29, for reasons that are explained later in this chapter section. The central component around which all other factors interact and influence is the teacher's pedagogical goals. The purpose of John's (2006) model is to assist preservice teachers to plan lessons by taking an iterative approach to lesson planning and moving back and forth between various factors. John (2006) notes, however, that inexperienced preservice teachers cannot manage all 29 factors when they first begin to plan lessons.

Of the 29 influencing factors, John (2006) contends that preservice teachers at the beginning of their teacher education program focus only on nine of those factors. John (2006) also posits that nine of the remaining 20 factors are only considered by experienced preservice teachers at the end of their degree program. Table 2 presents the 29 factors identified by John (2006), according to the preservice teachers' level of experience.

Table 2. Factors contributing to preservice teachers' preactive decisions (John, 2006)

Inexperienced preservice teacher	As experience is gained ------------------→	Experienced preservice teacher
Aims/objectives/Learning outcomes	-	-
Subject matter	Representations Depth and breadth	Conceptual understanding
National curriculum Schemes of work Unit plan	-	Cross curricular
Available resources Construction of resources	Choosing resources ICT, video, text	-
Task and activities	Type/level/material Usability	Degree of difficulty
Classroom control Behaviour	Class chemistry Routines and expectations	-
-	Students learning Age/ability	Learning styles Differentiation
-	-	Professional values Beliefs Inclusion Equal opportunities

Table 2 shows that as preservice teachers become more experienced, they are able to consider more factors as they make their preactive decisions, including class dynamics, beliefs, conceptual understanding, and depth and breadth of subject content. John's (2006) model suggests that preservice teachers will plan to teach lesson content more broadly and deeply as they gain experience teaching in the classroom. However, John's (2006) description of preservice teacher development does not make reference to increasing experience in teaching certain topics but simply to any teaching experience. For the participants in this study, although they had all experienced teaching mathematics in some or all of their practicums prior to the collection of data (details are provided in chapter 3), their experience in teaching algebra was far more limited. Hence, it is difficult to ascertain, using John's model, whether preservice teachers might be expected to emphasise conceptual understanding of a topic that they are teaching for the first time.

The second theoretical framework of teacher decision making reviewed is that of Schoenfeld (2010; 2011). Schoenfeld's initial theoretical model (Schoenfeld, 1999) was used to analyse teacher decisions in empirical studies of expert mathematics teachers (Aguirre & Speer, 1999; Schoenfeld, 2002; Schoenfeld, 2008; Schoenfeld, Minstrell, & van Zee, 1999; Sherin, Sherin, & Madanes, 1999) and preservice teachers (Zimmerlin & Nelson, 1999). Schoenfeld provides a refined theory of human in-the-moment decision making in his book, *How we think* (2010).

According to Schoenfeld's theory, all actions performed by a teacher are the result of decisions and it is by studying the decisions of teachers that the practice itself can be better understood. Within Schoenfeld's theory, he distinguishes between different levels of expertise in his description of how decisions are made and the quality of resources informing those decisions. Schoenfeld's theory focuses on decisions made during the interactive or teaching phase of instruction. Although he does make reference to preactive decisions, formed prior to a lesson, he does not elaborate on the process by which teachers make those decisions as other theorists do (e.g., Shavelson & Stern, 1981; Westerman, 1991).

Teachers' decisions to act are a function of three constructs, according to Schoenfeld (2010; 2011): goals, resources, and orientations. Schoenfeld (2010) contends that teaching is a goal-oriented activity and that every decision to act in a classroom situation is made with a particular goal in mind. The live decision making process begins with the formation of goals, either consciously or unconsciously, as the teacher familiarises him or herself with the classroom context in that moment. A pre-existing goal from the teacher's mental plan of the lesson, referred to as a "lesson image" (Schoenfeld, 2010, p. 24), may be re-established in-the moment or a new goal may be established if unforeseen events occur. The teacher decides upon a course of action to achieve the goal. That course of action is either a well-known set of actions, if the situation is familiar to them, or alternatively, a process of subjectively valuing the options available to them in order to decide upon the best set of actions for an unfamiliar situation. Schoenfeld (2010; 2011) suggests that as teachers familiarise themselves with a situation, form goals, and decide upon actions, they draw upon resources (knowledge and other intellectual, social, and material resources) and orientations (beliefs, values, preferences, and dispositions) to inform their decisions. The three constructs of goals, resources, and orientations succinctly capture a broad range of decision making elements, reflecting Schoenfeld's own goal of explaining "the choices people make in knowledge-intensive, highly interactive, dynamically changing environments" (p. 6) such as teaching.

Elements of Schoenfeld's model have been successfully applied to several studies of teachers' actions and decisions. The model has been applied generally to identify the goals, resources, and orientations that influence experienced teacher actions (Schoenfeld, 2002; Schoenfeld et al., 1999), graduate teacher actions (Stadler, 2011), and preservice teacher actions (Zimmerlin & Nelson, 1999). Individual constructs of the model have also been employed to serve as distinct foci through which to view and understand teacher decisions. Aguirre and Speer (1999), for example, focused on how beliefs shaped goal setting and prioritisation, whereas Sherin et al. (1999) studied the influence of knowledge content and form on decision making. The different applications of Schoenfeld's theory reflect the utility of the model in explorations of teacher decisions and actions.

One of the strengths of the model, according to Schoenfeld (2010), is that it can be used to analyse decisions that lead to actions at a macro or micro level. Schoenfeld (2010) contends that teachers' actions at a macro level, such as checking homework or presenting new lesson content, can be explained using the constructs of goals, resources, and orientations. He argues that micro level actions such as a single verbal statement can also be explained as a function of a teachers' (micro level) goals, resources, and orientations. Zimmerlin and Nelson (1999) found that studying teaching actions only at a fine grain level had a number of limitations. In a study of a secondary mathematics preservice teacher's actions and decisions, they found that the teaching practice could be better understood if broader action sequences were analysed to supplement the analysis of single actions. Schoenfeld later agreed with Zimmerlin and Nelson's (1999) observations, acknowledging that it is sometimes "difficult to see the forest for the trees" (Schoenfeld, 2010, p. 71) when single actions are the only unit of analysis. Schoenfeld's model can therefore be used to explain the decisions and actions of teachers and appears to do so most effectively when varying grain levels of analysis are employed. Consequently, the analytic approach taken in this study investigated the participants' MCK related decisions and actions at the macro, meso, and micro levels of their lessons.

Live interactions between teachers and students are crucial influences on teachers' decisions, according to Simon's (1995) decision making model, the *Mathematics Teaching Cycle* (p. 137). Simon's (1995) reflections of his own teaching as a mathematics teacher educator led him to design a model of interactive decision making, underpinned by a social constructivist perspective. He posits that it is the experience of interacting with students that is a key factor impacting the decisions made by teachers during a live lesson.

The decision making process is presented in Simon's (1995) model as a cycle of continual adaptation and updating of prior decisions. Preactive decisions draw upon several knowledge types, according to Simon (1995), including content knowledge and several aspects of PCK. One aspect of PCK (knowledge of content and students) is hypothesised by the teacher for the class in mind. The capacity for a teacher's thinking to be predictive or hypothetical in nature is captured in Simon's (1995) concept of a "hypothetical learning trajectory" (p. 136) which is the expected path that the teacher predicts the lesson will take.

The hypothetical learning trajectory includes three components that are created in the preactive phase of teaching and refined during the interactive phase of teaching. The first component is a learning goal the teacher is aiming to achieve. The second component is a plan of action, such as posing a particular problem, and the third component includes how the teacher expects the interaction to unfold and the predicted outcome of the interaction. During the interactive phase of teaching, the teacher begins with actions that form part of the learning trajectory. As new information is gleaned from interactions with students, the learning trajectory, including the goals, actions, and predicted outcomes, is modified as a result of new knowledge and hypothetical information improves. Interactions that unfold in a lesson allow for teacher knowledge to be modified during instruction and as that knowledge is refined, learning trajectories are improved and better informed decisions and actions should result. Drawing on Simon's (1995) contention that interactions can shape teacher knowledge, this study investigated the effect that live student interactions had on the knowledges held by the preservice teachers and the MCK they subsequently decided to enact as a result of those interactions.

Three of the earliest cognitive models related to teacher decision making form part of the framework created by theorists Shavelson and Stern (1981). The three models were conceptualised as a result of a large review of research on teachers' pedagogical judgements, decisions, and actions. The first model, which is a refinement of Shavelson's prior work (Shavelson, 1976), identifies teachers' pedagogical judgments as strong influences on the decisions they make and additionally, identify elements that contribute to those judgements. It is suggested by Shavelson and Stern (1981) that teacher judgements directly affect the decisions teachers make in the preactive and interactive phases of their teaching practice. The elements that impact those judgements included institutional constraints, teacher beliefs, and knowledge of subject content, students, and pedagogy. Although the model was not generated specifically for preservice teachers, the presence of institutional constraints suggests that the practicum context may influence preservice teachers' instructional decisions, a possibility that was explored in this study.

Shavelson and Stern's (1981) second cognitive model is a non-causal network of elements that inform teachers' preactive decisions. They found, after reviewing empirical studies of teachers' instructional planning, that teachers consider content, students, materials, activities, social community, and goals when they form their mental plan or image.

The third model created by Shavelson and Stern (1981) relates specifically to interactive decision making and emphasises the significant influence of classroom events on live teaching decisions. The process by which teachers decide to act once a lesson has begun is presented in the model. As the lesson takes place, Shavelson and Stern (1981) posit that the teacher seeks "observational cues" (p. 483) from students to inform their decisions on whether to continue as planned or to deviate from their plan. If an observational cue is judged to be outside an acceptable tolerance level for the teaching actions currently being implemented, the teacher must decide on a new course of action. The influence of live classroom circumstances on preservice teachers' MCK related decisions highlighted in this model and also in Simon's (1995) framework were investigated in this study.

Decisions to begin or end a sequence of teaching actions or a larger lesson segment within a single lesson are scrutinised in the cognitive framework created by Leinhardt and Greeno (1986). The researchers were interested in what effect the outcome of one action had on the in-the-moment decision making process that ensued as a teacher decides whether to continue an action sequence or to end it and begin a different one. Leinhardt and Greeno (1986) found that the outcome of one action offered the teacher new knowledge that they used to inform subsequent pedagogical decisions.

Leinhardt and Greeno (1986) studied the teaching practice of eight expert and four preservice primary teachers for three and a half months. They found teachers begin with a mental image of how they imagine the lesson will unfold, which they refer to as an "agenda" (Leinhardt & Greeno, 1986, p. 76). The lesson agenda is made up of envisaged lesson segments and sequences of teaching actions. The lesson agenda also includes goals associated with sets of teaching actions and the conditions for beginning, continuing, or terminating an action sequence or segment. They contend that as teachers enact the components of the lesson agenda, they gather knowledge that informs their future decisions about how to proceed in the lesson. For example, if a condition to end an action sequence is satisfied, such as students getting the correct answer or becoming bored with an activity, the teacher may proceed with the next lesson image component rather than continuing the sequence currently in progress. The sequence of gaining knowledge to inform future decisions continues throughout a lesson and decision making and teaching practice is therefore viewed as socially dynamic.

To illustrate the framework in practice, Leinhardt and Greeno (1986) described the process using several lesson excerpts. Fine-grained analyses of lessons using small sets of actions as the units of analysis were presented within broader descriptions of lesson segments and their associated goals. Each small set of actions was described and grouped with a corresponding function (i.e., pedagogical goal) and an outcome, if one occurred. The nesting of smaller sets of teaching actions and their associated pedagogical goals within broader sets of actions by Leinhardt and Greeno (1986) formed part of the analysis process undertaken in this study.

The final conceptual framework reviewed is the framework of preservice teacher decision making by Westerman (1991). The framework includes a model specifically designed to capture preservice teacher preactive and interactive decision making. The model was developed as part of the findings of an empirical study of five primary preservice teachers and their mentor teachers.

Westerman found that knowledge of content, students, and teaching are elements that are far less likely to shape preservice teachers' preactive and interactive decisions because they are not well understood by preservice teachers. It should be noted, however, that the preservice teachers in this study were primary preservice teachers and not those studying specifically to be mathematics teachers. Nevertheless, an implication of this research for the methodological design of this study was to explore whether deficiencies in preservice teachers' knowledge, and particularly their MCK, led to particular decisions where enacting MCK was concerned.

A second significant conclusion drawn by Westerman (1991) was that the preservice teachers' mental images formed prior to the lesson (i.e., during preactive decision making) influence live teaching actions most directly. The preservice teachers in Westerman's study rarely chose to modify their preactive decisions during instruction. Westerman (1991) suggested that the preservice teachers' reticence about adapting their lesson images after the lesson had begun was due to the nature of the goals they established as part of their preactive decisions. Westerman (1991) explained that preservice teachers form highly structured preactive goals, based on their beliefs and values, their curriculum knowledge, and broader lesson goals. The highly structured nature of preactive goal setting by preservice teachers is said to account for the unlikelihood of preservice teachers to respond to student cues and adapt their teaching actions to achieve modified goals. Westerman (1991) argues that the preservice teachers' desire to achieve the lesson objectives leaves little opportunity for them to make different decisions about their actions once teaching has begun. Hence, a second implication of Westerman's research for this study was to investigate conditions, if any, where secondary mathematics preservice teachers were hesitant about enacting particular MCK in response to student cues.

2.3.2 Influencing elements impacting preservice teacher decisions

Elements of the six decision making frameworks described in section 2.3.1 contribute towards a picture of how preservice teachers might make their decisions about the MCK they enact while teaching. In section 2.3.2.1, the process by which preservice teachers make preactive and interactive teaching decisions are discussed, to identify key components of instructional decisions that pertain to preservice teachers and MCK. In section 2.3.2.2, contributing factors of preservice teachers' decisions are explored to supplement the influencing elements identified in section 2.3.2.1.

2.3.2.1 Components of the preservice teacher decision making process

The processes by which preservice teachers make preactive and interactive decisions, according to the models and/or their accompanying descriptions in each conceptual framework, are summarised in Table 3. The level of detail provided in the table regarding the decision making process in the preactive and interactive phases differs according to the level of detail provided in the reporting of the processes in the frameworks. When specific distinctions were made by the researchers between expert and preservice teacher decision making in their descriptions of teacher decision making, only those aspects pertaining to preservice teachers were included in Table 3.

Two aspects of the decision making process common to all frameworks are evident in Table 3. Firstly, all theorists identify the presence of goals in their models but differ with respect to how rigid lesson planning goals remain during instruction. All theorists also agree that the decision making process relies, in part, upon forming sets of teaching actions or choosing from ready-made sets of teaching actions as a means to achieve instructional goals. The decision making process pertaining to the teacher's choice of goals and their choice of means to achieve their goals are now elaborated.

Preservice teacher intentions or goals that lie behind teaching actions are a critical facet of the decision making process. An assumption underpinning each framework and emphasised by Schoenfeld (2010) and Leinhardt and Greeno (1986) is that teachers are rational people who act with the intention of achieving one or multiple goals. When a teacher establishes an instructional goal, he or she chooses a particular action or set of actions to perform in pursuit of that goal (Leinhardt & Greeno, 1986; Schoenfeld, 2010). Hence, the teacher's desire to achieve a particular goal drives the teacher's decision to act.

In the context of teaching, goals are often pedagogical in nature, as can be seen in Simon's (1995) use of the term "learning goal" (p. 136) in his framework. Goals can also refer to control of the classroom environment, noted in Shavelson and Stern's (1981) framework. Given the complex nature of a live classroom, teaching actions can be performed to achieve multiple goals and the presence of multiple goals can lead to the teacher having to manage several competing goals at once (Schoenfeld, 2010; Shavelson & Stern, 1981). A teacher needs to prioritise certain goals over others at any given moment, then decide upon the best course of action to satisfy the prioritised goal(s) (Schoenfeld, 2010).

Table 3. Preservice teachers' decision making processes, before and during a lesson

Author (date)	Decision making before the lesson begins	Decision making during the lesson
John (2006)	The teacher moves back and forth between contributing factors to form a lesson image. The lesson image comprises goals and learning activities to achieve those goals. The teacher's experience determines the contributing factors that are considered. Context determines factors that are of most relevance.	The process is not included in the framework.
Westerman (1991)	The teacher forms a mental representation of the lesson (lesson image). The mental representation comprises highly structured preactive goals that often mimic lesson objectives and a mental plan of actions to achieve those goals. The mental representation is unlikely to include routines (well-practised action sequences).	Preactive decisions remain the priority. Structured goals from the preactive stage inform and constrain interactive decisions. Cues from students to modify planned actions are largely ignored.
Shavelson and Stern (1981)	The teacher forms a lesson agenda (lesson image). The lesson agenda comprises goals and routines (well-practised action sequences) or non-routine action sequences to achieve those goals. Mental scripts (mental plans of what will be said) are chosen or created as part of the action sequences. The process by which elements such as content knowledge, student knowledge, and activities are integrated to form the lesson agenda is not described.	Preactive decisions remain the priority. Routines are enacted wherever possible to reduce unplanned decision making, reduce cognitive effort and maximise lesson flow. As teachers seek cues (student behaviour) to evaluate success of routines, decision making occurs. Decision type 1: If cue is within tolerance, a decision to continue is made. Decision type 2: If cue is out of tolerance, a decision to implement a new action sequence (preferably routinised) is made. Decision type 3: A decision to avoid the disruption of a new action sequence can be made if out of tolerance cue can be dealt with later or ignored.

Table 3. (continued)

Author (date)	**Decision making before the lesson begins**	**Decision making during the lesson**
Leinhardt and Greeno (1986)	The teacher forms a lesson agenda (lesson image). The lesson agenda comprises goals and action schemas, to achieve those goals. Action schemas can be routines (well-practiced action sequences). Action schemas are chosen if they include: - Planned action; - Intended consequence that is expected to meet the goal; - Requisite conditions which it is expected can be satisfied.	Preactive decisions are continually updated and revised. Decisions about enacting action sequences depend upon classroom events. An action sequence begins if prerequisite conditions are satisfied. The action sequence continues if corequisite conditions are satisfied. The action sequence ends if postrequisite conditions are satisfied (e.g., goal is reached) or a new action sequence is needed.
Schoenfeld (2010; 2011)	The teacher forms a lesson image. The lesson image comprises long and short term goals, action sequences of varying grain levels to achieve those goals, and a prediction of interactions that will occur. Routines (well-practiced action sequences) are called upon if available. Decisions leading to formation of lesson image are a function of a teacher's goals, orientations, and resources.	Preactive decisions are continually updated and revised. The teacher takes in the situation. Contextual features of the situation trigger certain knowledge to be activated. Pre-existing goals/subgoals are reinforced or new goals are established If routines/sub-routines (familiar action sequences) are available, they are enacted. If a routine is unavailable, action options are considered (resources influence what actions are available). The subjective expected value of each action sequence is considered. (influenced by orientations) and a decision is made to act. The decision making process can begin again if the process is interrupted or not going to plan (new goals and/or actions to achieve the goal(s) are considered).
Simon (1995)	Teacher predicts a hypothetical learning trajectory (part of a lesson image) which consists of: - Learning goal; - Activity plan to achieve the goal; - Anticipated path that learning process will take. Preactive decisions are likely to be poorer than interactive decisions, because student knowledge is hypothetical.	Preactive decisions are continually updated and revised. Interactions with students cause modifications to teacher knowledge. Modified knowledge informs new learning goals, action plans, and hypothetical learning trajectories, leading to better interactive decisions and improved actions over time. The cycle continues as the interactions that result continue to inform decisions.

The changeability of goals during instruction is less agreed upon across the frameworks. Although Westerman (1991) identifies goals in her framework, she stresses that novices' preactive decision goals tend to be highly structured in accordance with lesson objectives and as such, do not vary considerably during instruction. Westerman's view contrasts with Simon's (1995) cycle of constant updating and improving of goals while teaching. The theorists' different points of view may be explained by the level of MCK and PCK held by the participants of these studies. Westerman (1991) was describing in her study inexperienced primary preservice mathematics teachers, who lacked strong knowledge of content and students. In contrast, Simon (1995) was describing his own decision making process, as an experienced mathematics teacher educator who would be expected to hold far more robust forms of knowledge about mathematics and students. Secondary preservice teachers who have completed advanced mathematics courses may have a stronger knowledge of mathematics than primary preservice teachers but they are unlikely to hold the mathematical knowledge of a more experienced mathematics teacher. Based on the frameworks, it is unclear whether secondary preservice teachers would hold firmly to their preactive decisions as Westerman's (1991) participants did or whether they would adjust their mathematics teaching goals and actions during a lesson.

Schoenfeld (2010) accepts both Westerman and Simon's views as possibilities. He contends that goals can remain, be altered, or even be replaced with newly established goals, while in the act of teaching. The strong presence of goals in each model and the lack of research regarding the degree to which secondary preservice teachers amend or replace their MCK related goals when teaching necessitated the examination of preservice teacher goals in this study.

Every decision making framework reviewed in this section indicates that actions are chosen by teachers as a means to achieve instructional goals. In this study, it was the MCK related means by which the preservice teachers pursued their goals that were of interest. Thus, the discussion about the means to achieve instructional goals is presented with respect to enacting MCK in preactive and interactive decision making.

Teachers choose the means by which goals are pursued well before a lesson begins. A significant feature of preactive decision making, highlighted in every framework, is the form of mental imagery that teachers produce when planning their lessons. Teachers create a mental picture of how they imagine the lesson will unfold and that mental image includes teachers' choices of potential teaching actions, informed by pre-existing goals (Schoenfeld, 2010; Westerman, 1991). The mental image of a lesson is a feature of each framework included in Table 3, albeit with different language choices. Schoenfeld (2010), for example, refers regularly to a "lesson image" (p. 24), while Westerman (1991) prefers the phrase "mental representation [of a lesson]" (p. 293) and Leinhardt and Greeno (1986) employ the term "agenda" (p. 76). Simon's (1995) description of a "hypothetical learning trajectory" (p. 133) which forms part of his decision making model and is described in Table 3 also encompasses the mental imaginings of the teacher. For the purposes of this study, Schoenfeld's term, "lesson image" will be used to describe the preservice teachers' mental approximations of how they imagined their lesson would unfold, with a particular focus on the MCK related teaching actions that they planned to perform.

At various points during a mathematics lesson, teachers can choose to perform unplanned teaching actions. They can do so because they have decided upon a new goal to pursue or because the actions they are performing no longer appear to be a suitable means for achieving a previous goal (Schoenfeld, 2010). This study investigated the participants' decisions that led them to perform previously unplanned MCK related teaching actions. The decisions were investigated to ascertain if the teaching actions were the result of newly established goals or if they were considered an alternative means by which to achieve pre-planned goals.

Notably, the frameworks explicitly or implicitly reveal the difficulties associated with preservice teachers choosing a means by which to achieve an instructional goal in comparison with more experienced teachers. Shavelson and Stern (1981) explain that teachers implement teaching routines, which are well-known, familiar actions sequences if they are available to access. Other decision making frameworks shown in Table 3 also refer to routinised sets of actions (Leinhardt & Greeno, 1986; Schoenfeld, 2011; Westerman, 1991). The theorists note the reduced cognitive load associated with choosing to implement a well-practised set of teaching actions rather than having to develop an unfamiliar teaching sequence. Westerman (1991) concluded that routines can be called upon by experienced teachers while teaching but are not possessed by preservice teachers.

Leinhardt and Greeno (1986) claim that the increased cognitive load associated with a lack of routines reduces a preservice teacher's capacity to deal with unplanned situations. Leinhardt and Greeno (1986) found in their study that preservice teachers lacked the routines of more experienced teachers and therefore found the interactive decision making more cognitively demanding than their more experienced peers. Although their findings refer to the establishment of class routines and not specifically to MCK related actions, the phenomenon is consistent with the findings of Borko and Livingston (1989), who studied two secondary preservice teachers' classroom actions. Borko and Livingston (1989) found that the preservice teachers struggled to respond with appropriate mathematical explanations to their students during instruction. Given the likelihood of preservice teachers having to make instructional decisions with few available teaching routines to call upon, decision making can be a cognitively challenging experience for them. It is possible that preservice teachers may instead draw on mathematical routines when planning and teaching lower secondary algebra lessons. This possibility is elaborated further in section 2.4.3.

2.3.2.2 Factors that contribute to preservice teachers' decisions

The preservice teacher decision making frameworks, introduced in the previous sections, refer to a number of factors that inform teacher decisions. Certain factors are agreed upon by researchers, whilst others are either absent or are perceived as being of little influence on preservice teacher decisions. The factors that contribute to teaching decisions, identified in each decision making framework, are presented in Table 4, according to their influence on preactive and interactive decisions. If no distinction was made about factors contributing to the two types of decisions, the factors have been repeated in both the preactive and interactive decision columns. Where information was available in the frameworks, major and minor influences are also noted.

Table 4 reveals that the teacher decision making frameworks share a number of similar contributing factors but with different emphases on what matters most in this complex process. Two factors are present in all frameworks: teacher knowledge and contextual features of the lesson. Teacher knowledge, and more specifically, content knowledge was identified as an element in each framework but the frameworks differed with respect to the level of influence that content knowledge was perceived to have in the decision making process.

Table 4. Factors that contribute to preservice teachers' decisions

Framework	Factors identified as impacting teacher decisions	
	Preactive decisions	**Interactive decisions**
John (2006)	*Preservice teachers with little/no experience:* Subject content knowledge Curriculum guidelines - National, schemas of work, unit plan Pedagogical knowledge, activities Classroom control, behaviour Resources - construction, availability Lesson context* As *teaching experience increases:* 19 more factors (see Table 2)	Not included in the framework
Westerman (1991)	*Major influence*: Lesson objectives Curriculum guidelines Teacher beliefs, values *Minor influence:* Subject content knowledge Knowledge of teaching, pedagogy Knowledge of students	*Major influence:* Instructional goals from preactive phase *Minor influence:* Knowledge of students Cues (student behaviour) Awareness of off task behaviour Disciplinary strategies
Shavelson and Stern (1981)	Subject content knowledge Knowledge of students Materials Activities Social community	Subject content knowledge Knowledge of students Pedagogical knowledge Use of heuristics Teacher beliefs, judgements Institutional constraints Cues (student behaviour)
Leinhardt and Greeno (1986)	Subject content knowledge Knowledge of lesson structure	Subject content knowledge Knowledge of lesson structure Classroom events
Schoenfeld (2010; 2011)	Resources (emotional; intellectual, including multiple knowledge types; physical) Orientations (Beliefs; values; preferences; dispositions)	Classroom situation Resources (emotional; intellectual, including multiple knowledge types; physical) Orientations (Beliefs; values; preferences; dispositions)
Simon (1995)	Subject content knowledge Knowledge of students Pedagogical knowledge Pedagogical content knowledge	Subject content knowledge Knowledge of students Pedagogical knowledge Pedagogical content knowledge Student interaction

** "Lesson context" was located in the supplementary text accompanying John's (2006) model of 29 influencing factors.*

The potential for teacher knowledge to be altered while in the act of teaching was another point of difference. Each decision making framework also features a reference to the live classroom context and its impact on teacher decisions. These two contributing factors are elaborated, preceding a final discussion of the contributing factors that were highlighted in some models but not others.

Multiple knowledge types were highlighted within the decision making frameworks. Their presence reflects the position of Rowland et al. (2009) that knowledge brought by teachers to the teaching situation underpins instructional decisions. The types of knowledge that featured in most or all frameworks were those related to subject content, students, pedagogy, and curriculum. For this study, which focuses on preservice teachers' decisions concerning MCK, the theorists' referrals to content knowledge were of particular interest and are further elaborated.

Content knowledge is evident in each decision making framework, however its location and influence on teacher decisions varies. Content knowledge is positioned only in the factors contributing to preactive decision making in Westerman's (1991) framework (see Table 4), in contrast to the influence of content knowledge on preactive and interactive decisions in the frameworks of Schoenfeld (2010), Leinhardt and Greeno (1986), Simon (1995), and Shavelson and Stern (1981). Further, Westerman (1991) concluded that preservice teachers' content knowledge is only a minor influence on those decisions made before a lesson begins because their content knowledge is weak. In the remaining decision making frameworks, content knowledge is considered as equally strong an influence on teacher decisions.

Westerman's (1991) finding that content knowledge plays only a very minor role in the decisions of less experienced teachers contrasts with the findings of Amador and Lamberg (2013). In a study involving three expert primary teachers and one novice (graduate) primary teacher, Amador and Lamberg (2013) applied Simon's (1995) *Mathematics teaching cycle* model to explore the teachers' planning and teaching decisions. They found that the novice teacher drew on her content knowledge to make interactive decisions. The novice teacher decided to reteach mathematical concepts or spend more time than what she had originally planned teaching certain concepts during lessons in response to student misunderstandings. This finding aligns with Simon's (1995) and Schoenfeld's (2010) claims that MCK can significantly influence interactive decisions because it can affect what teachers pay attention to during lessons. Hence, preservice teacher MCK was included in the analysis framework of this study as a significant contributing factor of secondary preservice teachers' MCK related decisions.

The potential for MCK to develop during instruction was implied in one decision making model but not in others. Simon's (1995) social constructivist perspective of teacher decision making allows for teacher knowledge, including MCK, to change as a result of live student-teacher interactions. In contrast, Schoenfeld (2010) contends that when a teacher takes in a situation, certain knowledge becomes salient but he makes no reference to a change in knowledge as a result of being in a particular situation. Leinhardt and Greeno (1986) and Shavelson and Stern (1981) indicate that as students are monitored during instruction, a teacher's knowledge of their students' understandings may improve. However, neither of these theorists suggest that MCK might also be refined. Westerman's model does not allow for the possibility of content knowledge being reconstructed in any significant way during a lesson because she did not find preservice teacher knowledge to influence interactive decisions. Although there is little consensus about a teacher's development of their own MCK as a result of teaching, the data were analysed in this study for evidence of any teacher knowledge, and particularly MCK, being developed during instruction.

The situated nature of teacher knowledge, including MCK, is captured in the five decision making frameworks that pertain to interactive decisions. Each of those frameworks refer to the influence of certain contextual features of a lesson on teacher decisions. The classroom events that occur within a live lesson lead teachers to make particular instructional decisions (Leinhardt & Greeno, 1986; Schoenfeld, 2010; Shavelson & Stern, 1981; Simon, 1995; Westerman, 1991). Classroom events influence the decisions teachers make by triggering decisions to continue on with actions planned prior to the lesson or to undertake new sets of teaching actions (Leinhardt & Greeno, 1986, Schoenfeld, 2010; Shavelson & Stern, 1981; Simon, 1995; Westerman, 1991). Studies of preservice teachers enacting MCK in live classroom situations, described earlier in this chapter, show how classroom events can influence preservice teachers' decisions and actions (Borko & Livingston, 1989; Livingston & Borko, 1990; Rowland et al., 2011; Thwaites et al., 2011). The analysis of data in this study therefore included an analysis of particular types of classroom circumstances that appeared to influence the participants' MCK related decisions and subsequent actions.

The review of the decision making frameworks revealed contributing factors that were not common to all frameworks. Leinhardt and Greeno (1986), for example, highlight knowledge of lesson structure as an element of decision making, a factor that is not explicitly named by the other theorists. Shavelson and Stern (1981), on the other hand, note the influence of institutional constraints on teachers' decisions. Given the preservice teachers were making decisions within a high stakes practicum context (Sim, 2011), the elements of the practicum context were considered a potentially significant influence on preservice teacher decisions in this study.

Teacher beliefs were explicitly identified as contributing factors for four of the decision making frameworks (John, 2006; Schoenfeld, 2010; Shavelson & Stern, 1981; Westerman, 1991) and were considered by Westerman (1991) as more influential than teacher knowledge for the preservice teachers studied. Although teacher beliefs did not feature in every decision making framework, they have been identified in other mathematics education literature as influencing mathematics teachers' actions.

Teachers' beliefs are a strong influence on their mathematics teaching practice (Barkatsas & Malone, 2005; Beswick, 2005; Cooney, Shealy, & Arvold, 1998; Raymond, 1997). Those beliefs can refer to the nature of mathematics, teaching mathematics, and learning mathematics (Artzt & Armour-Thomas, 1998; Ernest, 1989). The relationship between mathematical beliefs and instructional practice is an important one because teachers' beliefs about mathematics are communicated to their students when they teach (Goos, Stillman, & Vale, 2007) and are an influence on how mathematics teachers teach (Kinach, 2002a; Sullivan, 2003). Beliefs about mathematics teaching and learning also impact teaching actions (Rowland et al., 2011). For example, in a study of two practising mathematics teachers teaching lower secondary algebra lessons (Lewis & Blunk, 2012), the participants' beliefs about how students learn mathematics were found to have impacted the instructional actions they performed. Preservice teachers' beliefs about mathematics, mathematics teaching, and mathematics learning would therefore be expected to influence their decisions concerning MCK and were included as one of the potential influencing elements in this study.

2.3.3 Potential elements impacting preservice teachers' MCK related decisions

This major chapter section has identified a number of influencing elements which may influence secondary mathematics preservice teachers' MCK related decisions during instruction. Figure 1 synthesises the key ideas presented in the six frameworks into a model of influencing elements that the study operationalised for the analysis of the participants' MCK related decisions. The elements comprise components of the decision and factors contributing to the decision.

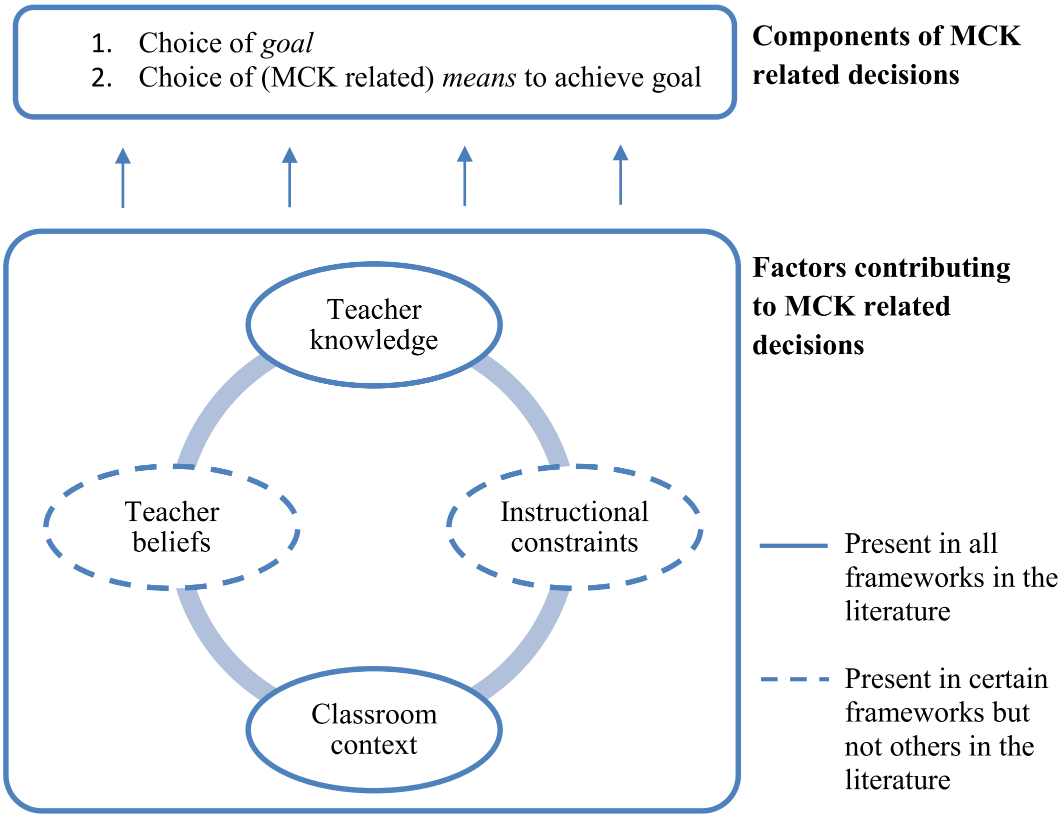


Figure 1. Potential elements impacting preservice teachers' MCK related decisions

As shown in Figure 1, MCK related decisions comprise two major components in this study: choosing and prioritising goals and deciding upon the MCK that should be delivered to achieve those goals. The complexity of the decision making process is evident in Figure 1 in the four contributing factors linked together, showing how preservice teachers may simultaneously consider multiple factors when deciding on goals to pursue and MCK to enact during instruction. The lack of teaching routines available to preservice teachers when they choose MCK to deliver means that the process is a cognitively demanding one. The review of teacher decision making frameworks indicated that multiple knowledges and contextual features of the classroom are very likely to feature in preservice teachers' MCK related decisions. Teacher beliefs and institutional constraints, which were not present in all frameworks, may also influence preservice teachers' MCK related decisions.

Hence, to study the MCK related actions of secondary preservice teachers, one knowledge type (i.e., MCK) cannot be successfully isolated or removed for analysis without consideration of the other elements influencing the decision making process simultaneously. That is why in this study, visible manifestations of MCK were not investigated in isolation but were studied with respect to the equally significant invisible aspect of enacting MCK, that is, the preservice teachers' decisions and the array of elements that impact those decisions.

2.4 Experiences impacting the quality of preservice teachers' MCK related decisions

In section 2.3, the literature pertaining to teacher decision making and where possible, preservice teacher decision making was reviewed and synthesised. Two sets of influencing elements, shown in Figure 1 (section 2.3.3), concern the knowledge and beliefs that preservice teachers bring to the decision making process. These sets of influences warrant further investigation in the literature for two reasons. Firstly, preservice teachers' knowledge and beliefs about mathematics and mathematics teaching develop first in their experiences as mathematics learners. Secondly, their knowledge and beliefs are yet to be significantly refined as a result of mathematics teaching because they lack teaching experience. The final major section of this chapter reviews the literature concerning preservice teachers' experiences as mathematics learners and teachers and how these experiences might impact their MCK related decisions.

2.4.1 Preservice teachers' own schooling experiences

The tendency for teachers to teach in the way they were taught has been widely acknowledged (Fajet, Bello, Leftwich, Mesler, & Shaver, 2005; Holm & Kajander, 2012;

Llinares & Krainer, 2006; Lortie, 1975; McNeal & Simon, 2000; Prescott & Cavanagh, 2006). All teachers were once students themselves and their years of observing and interpreting their own teachers' behaviours lead them to form theories of what it means to be a teacher (Fajet et al., 2005). For example, Prescott and Cavanagh (2006) interviewed 16 Australian secondary mathematics preservice teachers about their memories of school mathematics, their reasons for becoming a mathematics teacher, and their beliefs about mathematics teaching. They found that the preservice teachers' own schooling experiences shaped, to a large extent, the beliefs they held about the nature of mathematics and mathematics teaching. Other studies have found that those beliefs influence preservice teachers' MCK related decisions and actions (Rowland et al., 2011; Westerman, 1991).

Basing one's beliefs about how one should teach on prior schooling experiences is problematic because of the student vantage point from which that teaching practice was viewed. Lortie's (1975) claim that future teachers complete an "apprenticeship of observation" (p. 61) during their schooling is well cited in education literature. Lortie states that "participation in school has special occupational effect on those who do move to the other side of the desk" (1975, p. 61) because students form an idea of what it means to be a teacher by observing the teachers' classroom actions and interpreting those actions using a student's point of view. The student perspective of the work of teachers is limited according to Lortie (1975) because aspects of teachers' work such as goal setting, preparing lessons, pedagogical considerations, and reflections on classroom events do not form part of the apprenticeship. Hence, it is unlikely that the immature beliefs preservice teachers possess about mathematics or mathematics teaching as a result of their schooling are ideally suited to teaching decisions. It is also possible that an apprenticeship of observation fails to prepare preservice teachers to manage multiple students' needs in a live classroom context because, for secondary mathematics preservice teachers, the apprenticeship is completed from a personal and usually a successful mathematics student perspective.

Only some aspects of mathematical knowledge are developed by school students as a result of their schooling apprenticeships. Success in secondary mathematics would reflect a degree of knowledge but not necessarily MCK needed for the work of teaching. School students would not be aware of certain types of content knowledge specific to the work of teaching, such as horizon knowledge, so that MCK would be absent or underdeveloped.

Secondary school students may not hold a comprehensive knowledge of all secondary mathematics ideas at the end of their schooling. Engelbrecht (2008) claims that secondary school mathematics classes focus predominantly on mathematics as a set of algorithms, completed in response to key words, rather than as a way of thinking. If this is the common type of mathematical experience for preservice teachers, they may not have well developed knowledge of mathematical concepts, procedures, or mathematical ways of thinking, including those associated with algebra, when they begin their tertiary study. Unfortunately, although secondary mathematics preservice teachers complete tertiary advanced mathematics courses as part of their teacher education programs, Ball (1990) and Shoaf (2000) argue that participating in those courses does not give preservice teachers the opportunity to revisit and develop their understandings of the secondary school concepts they will be teaching. Therefore, the mathematical understandings they have developed about lower secondary algebra during their schooling may persist alongside underdeveloped ideas of mathematics teaching, without further development, in their own teaching practice.

2.4.2 Preservice teachers' tertiary education experiences

Secondary preservice teacher education programs typically include undergraduate mathematics courses, general education courses, mathematics education courses, and school-based practicums (Tatto et al., 2010). Preservice teachers develop knowledge and beliefs about mathematics, mathematics pedagogy, and themselves as teachers in all of these learning situations (Peressini, Borko, Romagnano, Knuth, & Willis, 2004). This section explores how preservice teachers' knowledge and beliefs, particularly regarding their MCK, develop as a result of their experiences in undergraduate advanced mathematics courses, general and mathematics specific education courses, and school-based practicums.

2.4.2.1 Advanced mathematics courses

Historically, the mathematical knowledge component of the secondary teacher education degree has been provided via advanced mathematics courses, usually taught by mathematicians in undergraduate programs. Courses such as these provide advanced mathematical knowledge to preservice teachers who are part of a cohort that includes students of science, engineering, mathematics, and technology (Zazkis & Leikin, 2010). This is still the case in Australia (Tatto et al., 2010). According to Ferrini-Mundy and Findell (2001), the assumption that the study of advanced mathematics content was the best preparation for preservice teachers as far as MCK is concerned has been held by teacher educators for several decades. It was also assumed that mathematics preservice teachers would benefit from seeing how the mathematics content they would later teach at a secondary level develops for various vocations. This broadened view was intended to improve the way preservice teachers might later present mathematics to their students to prepare them for future careers (Ferrini-Mundy & Findell, 2001). Research into the effectiveness of this approach to teacher education has produced less than convincing results.

There is evidence to suggest that success at university level mathematics does not necessarily translate into a strong fundamental understanding of school mathematics for preservice teachers (Wilburne & Long, 2010). Research by Begle (1979, cited in Speer & Hald, 2008), Cooney, Wilson, Albright, and Chauvot (1998), and Harris and Sass (2007) revealed no positive correlation and in some instances revealed a negative correlation between the number of advanced mathematics courses completed by a teacher and the subsequent student achievement levels of their students. Preservice teachers developing knowledge and beliefs about mathematics and mathematics education that are not well suited to teaching have been attributed, at least in part, to the advanced mathematical knowledge approach to initial teacher education programs.

Researchers suggest that taking advanced mathematics courses lead preservice teachers to develop compressed forms of the mathematical content they will eventually teach. It has been suggested that advanced mathematical knowledge requires an "increasing compression of knowledge that accompanies increasingly advanced mathematical work" (Ball, Lubienski, & Mewborn, 2001, p. 442). Teachers may therefore find it difficult to unpack the content needed for secondary school level mathematics or to articulate the connections between mathematical content of differing levels of complexity (Hodge, Gerberry, Moss, & Staples, 2010).

A deep understanding of secondary school mathematics tends not to develop in advanced mathematics courses (Ball, 1990; Shoaf, 2000). The reporting of secondary preservice teachers' mathematical knowledge earlier in this chapter (section 2.2) revealed a lack of conceptual understanding in many studies involving secondary mathematics topics, even though the participants would have completed advanced mathematics courses as part of their program. Preservice teachers are unlikely to enact conceptually strong MCK of algebra if that type of MCK is not a resource for them to draw on when making instructional decisions and the completion of advanced mathematics courses appears to do little to develop this important type of knowledge.

The development of desirable AWOTS as well as conceptual knowledge might be missed in advanced mathematics courses. Rasmussen (2001), for example, studied six tertiary students' understandings of differential equations using interviews featuring "think aloud" (Ericsson & Simon, 1993, p. 61) protocols. He noted two undesirable ways of thinking that were present in the tertiary students' mathematical actions: "mindless symbolic manipulation" and "mindless graphical manipulation" (Rasmussen, 2001, p. 67). Rasmussen (2001) maintains that these types of mathematical ways of thinking are consistent with rule based mathematical cultures that students experience in secondary mathematics and tertiary advanced mathematics courses.

Holding impoverished MCK may have a negative influence on the quality of MCK that preservice teachers know to enact if steps are not taken to improve MCK before teaching. MCK that was limited in some way was noted in studies of secondary preservice teachers' classroom actions (Livingston & Borko, 1990; Rowland et al., 2011), reported in section 2.2.2. An implication of limited MCK may be that as well as enacting poorer versions of MCK, preservice teachers might also avoid enacting particular MCK altogether if they do not feel confident in mediating certain mathematical ideas with students in a practicum lesson, particularly during unplanned conversations with students. They may instead choose to present only those aspects of their MCK that they feel most confident of delivering successfully. Poor MCK may therefore impact the process of MCK related decision making and may also be the product of that decision for preservice teachers. It was for this reason that the framework developed to analyse the MCK enacted in this study allowed for the presence of limited forms of MCK, such as compressed MCK and impoverished MCK, where procedures, concepts, and ways of thinking were concerned.

Preservice teachers develop beliefs about the nature of mathematics and mathematics teaching from their advanced mathematics experiences. However, like their schooling experiences, those beliefs are unlikely to be ideal for effective mathematics teaching. In a study of 173 secondary preservice mathematics teachers, Goulding, Hatch, and Rodd (2003) found that most participants viewed their undergraduate mathematics experiences as focusing only on rote learning of meaningless work. Their impoverished views of mathematics led them to "relinquish… the ambition to understand" (Goulding et al., 2003, p. 385) the mathematical content of the course, revealing poor beliefs about mathematics that should not be communicated to students. Their repeated exposure to mathematics teaching in a lecture format, the traditional instructional format in advanced mathematics courses (Morrel, 1999; Speer, Smith, & Horvath, 2010) can lead them to teach as they were taught and see mathematics teaching as presenting compressed rules to be remembered for a test (Goulding et al., 2003). The potential for preservice teachers to hold unfavourable beliefs about mathematics and/or mathematics teaching and learning as a result of their advanced mathematics experiences was therefore considered in the development of the analysis framework for this study.

2.4.2.2 General and mathematics education courses

As part of their teacher education program, secondary preservice teachers complete courses in general education and mathematics education. General education courses comprise a significant proportion of the program and include courses on child development, learning theories, pedagogy, and educational practice (Lawrance & Palmer, 2003). Preservice teachers, however, need to acquire in their tertiary education pedagogical knowledge that refers specifically to mathematics teaching (Ball et al., 2008; Goos, 2013; Graeber, 1999; Kinach, 2002a; Tanisli & Kose, 2013). Stotsky (2006) argues that pedagogical skills that are particular to subject areas, including mathematics, cannot be adequately developed in general education courses. She contends that teachers are unlikely to develop discipline specific pedagogical knowledge when they are presented with generic teaching strategies because they must then work out how and when to apply that strategy to their own content area. PCK for teaching secondary mathematics is more likely to be developed in mathematics education courses.

In Australia, and for the participants in this study, mathematics education courses make up a very small proportion of the courses completed by secondary preservice teachers in their undergraduate programs (Lawrance & Palmer, 2003). However, the responsibility for developing adequate PCK for teaching mathematics falls to the teacher educators of these courses. MCK for teaching also needs to be developed in mathematics education courses because advanced mathematics courses focus very little on secondary mathematics topics (Borko et al., 1992; Hodge et al., 2010).

There is evidence that participation in mathematics education courses can bring about positive changes in preservice teachers' MCK, PCK, and their beliefs about mathematics and mathematics teaching and learning (Davis, 2009; Kinach, 2002a; Kinach, 2002b). Davis (2009), for example, found that preservice teachers' participation in a mathematics education course brought about improved understandings of exponential functions (MCK) and a greater knowledge of potential student misconceptions regarding exponents (PCK). Kinach (2002a), a secondary mathematics teacher educator, offered her preservice teachers' in class opportunities to confront their beliefs about mathematics teaching, which centred initially around mathematics teaching as "a show-and-tell process" (Kinach, 2002a, p. 64). Over time, Kinach's (2002a) preservice teachers' beliefs about mathematics teaching began to include references to developing students' understanding of mathematical content, reflecting the positive influence that mathematics education courses can have on preservice teachers' knowledge for teaching. Nevertheless, the very small allocation of courses of this type in comparison with advanced mathematics and general education courses limits preservice teachers' opportunities to develop their knowledge and beliefs about mathematics and mathematics teaching that will influence their MCK related decisions.

2.4.2.3 The school-based practicum

The practicum offers preservice teachers the opportunity to learn about mathematics teaching practice as members of the teaching community and in live teaching contexts. Lave and Wenger (1991) contend that within a "community of practice" (p. 98), members of a community engage in activities that are underpinned by common "understandings of what they are doing and what that means…for their communities" (p. 98) and it is through participating in communities of practice that learning takes place. During the practicum, preservice teachers participate in mathematics teacher communities of practice in the schools to which they are assigned. Their participation within the community of practice offers them more than the limited vantage point of observing teaching that they have as school or even university students. Instead, they gain access to a wider range of valued activities related to the mathematics teaching profession and perform those activities with other practising teachers (Cavanagh & Prescott, 2007). As their participation increases, they become more absorbed into the culture of the community (Cavanagh & Prescott, 2007) and over time, their knowledge and beliefs about mathematics and mathematics teaching tend to align with those of that community.

Preservice teachers in Australia have relatively limited experiences in school settings, compared with other countries (Tatto et al., 2010). This is not an ideal situation because, according to Lave and Wenger (1991), it takes time to gradually learn the practice of a community. Preservice teachers in Australia are therefore limited in their practicum opportunities to learn to be effective mathematics teachers.

The supervising teacher's advice can greatly influence preservice teachers' practicum experiences and learnings about mathematics teaching (Cavanagh & Prescott, 2007; Rowland et al., 2009). Student teachers can improve their teaching using feedback provided by their supervising teachers (Rowland et al., 2009). For their MCK related actions to improve, feedback on this specific area of their teaching must be offered. Lewthwaite and Wiebe (2012) posit that supervising teachers may offer a benign (p. 49) practicum experience to preservice teachers by failing to provide detailed feedback on lessons or how preservice teachers might take steps to improve their instructional practice. Alternatively, feedback need not be related to lesson content, particularly if supervising teachers are out-of-field teachers themselves. Feedback may instead pertain to other issues, such as behaviour management or class routines, as the following example illustrates.

Leatham and Peterson (2010) studied 45 secondary mathematics supervising teachers' perceptions of the purpose of practicums in the United States. The researchers found that supervising teachers offered purposes that suggested the practicum experience was about preservice teachers learning about classroom management and having the opportunity to interact with a real teacher and real students. There was a notable absence of purposes relating to the preservice teachers learning how to teach lesson content or assist student learning. Further, the study findings also suggest that supervising teachers tend not to see themselves as teacher educators who should be providing learning opportunities to preservice teachers but instead as mentors whose role is to help the practicum run smoothly for the preservice teachers. The findings of Leatham and Peterson's (2010) study suggest that the development of preservice teachers' knowledge of and beliefs about mathematics, mathematics teaching, and mathematics learning may not be viewed by supervising teachers as their responsibility. This greatly reduces the potential for the preservice teachers' knowledge and beliefs to develop significantly during practicums as a result of their interactions with supervising teachers. A lack of MCK related feedback does not help preservice teachers to determine how effectively they are catering for the mathematical needs of their students when making instructional decisions.

When feedback related specifically to mathematics content is offered by supervising teachers, it does not always lead preservice teachers to enact high quality MCK. In the study findings of Markworth et al. (2009), reported earlier in this chapter, reflective conversations with a supervising teacher contributed to one secondary mathematics preservice teacher developing and enacting stronger MCK and PCK during the practicum. In contrast, Cavanagh and Prescott (2007), who interviewed eight secondary mathematics preservice teachers about their practicum experiences, found that their supervising teachers' advice encouraged limited MCK and PCK to be enacted. The preservice teachers in the study reflected that traditional approaches to mathematics pedagogy (e.g., teacher exposition, worked examples, and independent practice) and an emphasis on procedural mastery of textbook exercises were encouraged by their supervising teachers. The participants reported a compulsion to follow their supervising teachers' advice so they would receive a positive practicum report.

A similar finding was noted in Holm and Kajander's (2012) study of five primary preservice teachers' reflections of their mathematics practicum experiences. The researchers found that the practicum context emphasised forms of mathematics learning that included memorisation and procedural approaches to mathematics, appearing to restrict the preservice teachers' desire to teach more conceptually. The findings of these studies reflect Lave and Wenger's (1991) premise that the involvement of more experienced community members in the apprenticeship of newcomers can vary across different communities of practice.

In summary, the practicum, including the supervising teacher, has the potential to broaden preservice teachers' perspectives on teaching from the limited viewpoint of a student. Teacher knowledge, including MCK, and teacher beliefs can develop as a result of practicum experiences and will influence MCK related decisions. How those knowledges and beliefs develop, however, depend on how influential those practicum elements are in the experience of the preservice teacher.

2.4.3 Disparity between experience as an algebra teacher and performer

Preservice teachers have very little experience in teaching lower secondary algebra, compared with their many years of experience studying the discipline in their secondary and tertiary mathematics studies. Using the literature pertaining to novice and expert knowledge, the final discussion in this chapter explores the dual nature of novice and expert knowledge that preservice teachers hold with respect to teaching algebra. Holding both novice and expert knowledge can affect how preservice teachers might respond to the live algebra classroom context and how they decide upon the MCK to enact during instruction.

The novice to expert continuum provides important distinctions between the actions of novices and their more experienced counterparts. The continuum, first introduced by Dreyfus and Dreyfus (1980), then adapted for a nursing context by Benner (1982) and for a teaching context by Berliner (2001), comprises five stages of expertise, from novice to expert. At first glance, preservice teachers would be classified as pre-novices, given their inexperience in the classroom. However, their experiences over many years as successful mathematics learners must also be considered because they may have developed particular expert tendencies regarding certain areas of mathematics, including algebra. That means that it may be possible for preservice teachers, who are lacking in teaching experience, to draw on the expert knowledge they have from another mathematics education context, i.e., as a learner, to inform their teaching decisions. Therefore, descriptions of both novice and expert knowledge and skills are presented in this section with respect to secondary preservice teachers, who are pre-novice algebra teachers but expert performers of lower secondary algebra.

A novice (stage 1 of the novice to expert continuum) can decompose an environment, according to Dreyfus and Dreyfus (1980) into a number of context-free features, discernible by an inexperienced person. From those features, a small number of context-free rules are generated (Dreyfus & Dreyfus, 1980). Novices typically implement actions based on those context-free rules because they are unable to take in the contextual elements of a situation and respond to those elements (Benner, 1982; Berliner, 2001;

Dreyfus & Dreyfus, 1980). Over time, and with more experience in a variety of experiences, one becomes more adept at taking in the contextual features of a situation. A "competent" teacher (stage 3 of the continuum), for example, makes more instructional decisions based on particular classroom contexts and groups of students (Berliner, 2001). After a number of years, practitioners in a field may become experts (stage 5 of the continuum). Berliner (2001) suggests that teachers would take at least five years to develop into expert teachers. Expert practitioners become quite intuitive, responding automatically and instinctively to the contextual factors facing them in a given situation (Benner, 1982; Berliner, 2001; Dreyfus & Dreyfus, 1980). The expert does not think consciously about a situation or his or her performance and appears to perform effortlessly the right task at the right time (Berliner, 2001; Dreyfus & Dreyfus, 1980).

Emergent pedagogical skills are expected for preservice teachers who are yet to be considered even at the novice stage of the teaching profession, where a basic knowledge of teaching strategies should be known. As pre-novices, preservice teachers would not be expected to be able to respond effectively to live classroom circumstances and would rely on relatively context-free MCK related instructional actions. This description of pre-novice behaviour aligns with Borko and Livingston's (1989) descriptions of secondary preservice teachers during instruction who found it difficult to respond to their students' live contributions with adequate mathematical explanations. It is not until teachers gain more experience in teaching and progress to the more advanced stages of the novice to expert continuum that they are able to meld their acquired knowledge of typical events in situations with their knowledge of rules and procedures (Benner, 1982; Berliner, 2001). Only then does context become a more influential factor (Berliner, 2001). Preservice teachers, then, may not cope effectively within a live mathematics classroom context where unexpected events, requiring particular MCK to be delivered, must be dealt with as they transpire.

For preservice teachers, their years of experience in applying less sophisticated mathematics ideas in more advanced secondary and tertiary mathematics topics would elevate their skills and knowledge of lower secondary mathematics content to that of a developing expert (Nathan & Petrosino, 2003). One advantage of holding expert knowledge of lower secondary algebra is that mathematical routines are available to the preservice teacher to call upon when deciding on the MCK to enact in a lesson. Available routines are identified as reducing the cognitive load of teachers (Leinhardt & Greeno, 1986) so the preservice teachers' access to mathematical routines would appear to be advantageous. However, the development of expert knowledge of algebra including algebraic routines over a number of years prior to teacher education presents a challenge to future mathematics teachers.

MCK first develops outside the realms of a mathematics teaching perspective. Teachers initially develop and put into practice their mathematical knowledge as school students (Ball, 1990), and later as tertiary students (Zazkis & Leikin, 2010), mathematically minded citizens, and perhaps as other mathematically competent professionals. Consequently, MCK begins to develop well before a teaching perspective is introduced. Adler (2005) contends that the mathematics that teachers deliver is not synonymous with the mathematics used in other mathematics contexts. An implication is that difficulties may arise as a consequence of teachers learning mathematics content as non-teachers first and teachers second, resulting in two issues that significantly affect the immediate utility of MCK by a teacher. These issues are to do with the compressed form of mathematical knowledge that naturally develops over time and the de-emphasis placed on the verbal form of that knowledge.

Mathematical knowledge, as it develops over time, becomes more and more compressed and as noted previously in this chapter, occurs as a result of learning advanced mathematics (Ball et al., 2001). Compressed mathematical knowledge is recognised as an attribute of advanced mathematical competence because of the reduced cognitive load on the working memory. As the working memory is limited in capacity (Fetherston, 2007), aspects of knowledge, referred to by Miller (1956) as "chunks" (p. 93), can be grouped together. Miller claims that chunks of knowledge are needed so that larger amounts of knowledge can be held for mental use at any one time. Miller (1956) states that "since the memory span is a fixed number of chunks, we can increase the number of bits of information that it contains simply by building larger and larger chunks, each chunk containing more information than before" (p. 93). As an example in a mathematics context, Hiebert and Le Fevre (1986) suggest that "procedures are hierarchically arranged so that some procedures are imbedded in others as subprocedures and these subprocedures can be sequenced as an entire sequence, known as a superprocedure" (p. 7). They explain that connections between subprocedures and superprocedures allow for the retrieval of a sequence of subprocedures when one superprocedure is accessed, reducing the cognitive load required for more advanced mathematical work.

Miller's concept of a knowledge chunk is considered by many scholars as advantageous for students to develop. Kilpatrick et al. (2001), for example, encourage the formation of "knowledge clusters" (p. 120) which they identify as compacted groups of related mathematical ideas. Harel and Kaput (1991) suggest the development of "conceptual entities" (p. 82) to alleviate working memory, which they argue facilitates understanding of more complex mathematical ideas. Sfard's (1991) notion of reification and Gray and Tall's (1994) description of a procept also reflect a positive view of compression of mathematical knowledge. As secondary preservice teachers develop advanced mathematical knowledge, having progressed through secondary and tertiary mathematics courses successfully, a degree of compressed mathematical knowledge would be expected. However, good teachers need to have their knowledge in an accessible form (Ball & Bass, 2000; Adler & Davis, 2006) to draw upon as needed, so compression is not an ideal state in which teachers should hold their mathematical knowledge.

Unpacking teachers' expert algebraic knowledge is a difficult task because elements have been compressed over many years to cope with the demands of advanced mathematics. Maddox (1993), for example, says that "we take our system of numerical notation so much for granted that we are hardly aware of its characteristics … [such as] the use of position to indicate units, tens, and hundreds" (p. 71). This lack of mathematical awareness is problematic for teachers who need to be cognizant of mathematical features if they are to be able to teach them to students. Cohen (2011) contends that unpacking what has already been compressed into a more elegant and finished form is difficult because "the more polished our performances become in any realm, the less we remember how we did things less capably" (p. 112); yet, this is exactly what teachers must somehow accomplish. Expert knowledge of algebra, which preservice teachers may possess, potentially means that during instruction, preservice teachers may not be aware of all aspects of algebraic MCK they need to explicitly share with their students.

Possessing expert mathematical knowledge can lead to a second impediment to effective mathematics teaching, that is, a decreased ability to provide comprehensive mathematical explanations. Anderson (1980) explains that as facility in performing any skill increases, "verbal mediation in the performance of the task often disappears at this point. In fact the ability to verbalise knowledge of the skill can be lost altogether" (p. 235). For those preservice teachers possessing expert knowledge of lower secondary algebraic procedures, their ability to *perform* the mathematics may not be matched by an equally strong ability to *explain* the mathematics. This may be because mathematical knowledge is designed to be used, rather than taught, according to Davis (2008a). Davis maintains that elements of mathematics can be known in a tacit sense so they can be "simultaneously known (in the sense that they are enacted) and unknown (in the sense that they may not be available for conscious representation)" (Davis, 2008a, p. 2). If preservice teachers hold expert knowledge, they may hold it only in tacit form and may find it difficult to verbalise the mathematics they can so easily perform.

Preservice teachers of lower secondary algebra are therefore pre-novice teachers but may also be expert algebraic performers. This study investigated the influence that their mathematics experience and teaching inexperience had on preservice teacher MCK decision making and on the nature of the MCK enacted.

2.5 Conclusion

In summary, this chapter was positioned at the intersection of two fields of research pertaining to mathematics teaching: MCK and decision making. A synthesis of the literature that describes the MCK that teachers should ideally enact was presented as a benchmark against which preservice teachers' enacted MCK might be compared. Next, a review of studies that investigated preservice teachers' MCK was presented. The review revealed a need for studies such as this one to examine in detail the MCK that preservice teachers enact within a live classroom context. It also revealed that preservice teachers enact MCK of differing type and quality across lessons and in response to particular classroom circumstances.

The latter half of the chapter focused on the invisible aspects of enacting MCK in the classroom, the decisions that lead to MCK delivery. Decision making frameworks relating to novice and pre-novice teachers were compared and contrasted to highlight the process of decision making and potential influencing elements. The findings of the review were synthesised to produce the analytic framework for this study that was operationalised in the data analysis phase. Drawing upon the elements noted as influencing preservice teachers' instructional decisions, the chapter concluded with a discussion of secondary mathematics preservice teachers' educational experiences and the possible impact of those experiences on the quality of their MCK related decisions.

Chapter 3: Methodology

3.0 Introduction

The purpose of this study is to explore how preservice teachers enact mathematical content knowledge (MCK) of algebra in their practicum and the factors that influence their decisions to do so. To reiterate, the research questions are:

1. What elements influence the decisions secondary preservice teachers make regarding the mathematical content knowledge (MCK) they enact when teaching lower secondary algebra?
2. What is the mathematical content knowledge (MCK) that secondary preservice teachers enact when teaching lower secondary algebra?

This chapter begins with the theoretical perspective taken by the researcher that informed the overall design of the study. Elements of the methodological approaches that were used in the study design are then detailed, followed by a description of the participants and the context of the study. The research methods used to collect and interpret the data are outlined, including the analysis framework used to investigate enacted MCK and decision making elements concurrently. Finally, considerations regarding ethics, the researcher's role, and the limitations of the study are discussed.

3.1 Theoretical foundations

The design of the study was underpinned by the researcher's perspective of knowledge and the context in which it is enacted. These perspectives informed the methodological choices made regarding data collection, analysis, and interpretation.

A fallibilist philosophy of enacted MCK, where knowledge is considered impermanent and context-dependent, underpins this research project. This philosophy recognises that even if our justification for a particular belief is strong, we cannot guarantee that what we believe is true (Reed, 2002). Knowledge is regarded from a fallibilist perspective as a human construction and is therefore cognitively fragile and impermanent (Honderich, 1995). Ernest (1991) encourages educators to embrace a fallibilist philosophy of mathematics which acknowledges the fragility and uncertainty of the knowledge we hold. For preservice teachers, whose mathematical and pedagogical knowledge may be undergoing significant transformations as a result of their teacher education experiences, the impermanent nature of knowledge that underpins this study was an important methodological consideration in the model of decision making that was operationalised in the data analysis phases.

The context sensitive view of mathematical knowledge was a central premise of this study, well captured in Lave and Wenger's (1991) notion of "knowledge-in-practice" (p. 95) and von Glaserfeld's (1984) description of constructivism. Knowledge is situated, according to Lave and Wenger (1991) within the "cultural and political life of the community" (p. 100) in which it is located and is known "by specific people in specific circumstances" (p. 52). The fallibilist view does not go so far as to claim that all propositions we believe are false, as skepticism might (Lemos, 2007), but instead allows for what Cohen (1988) refers to as "relevant alternatives" (p. 94), where alternative propositions can be considered relevant and reasonable in certain circumstances. Conceptualising enacted MCK as situated knowledge required the researcher to consider the contextual location of the preservice teachers' knowledge and the conditions which produced awareness in a given moment (Brown, Collins, & Duguid, 1989; Maher & Tetreault, 1993; Mason & Spence, 1999; Stinson, 2004).

The impermanent and context-dependent nature of preservice teacher knowledge led to two general implications for the design of the study. The first was that the methodology needed to place the classroom context and more specifically, live teaching acts, at the centre of data collection and interpretation. Knowledge becomes, at any given time, what the practice has made it (Orlikowski, 2002), highlighting the importance of capturing the live experience of the preservice teacher to better understand knowledge-in-practice. Attempts to study knowledge away from the context of interest would have resulted only in speculation as to what might happen in the classroom. Lave (1988) warns that "to view the mind as easily and appropriately excised from its social milieu for purposes of study denies the fundamental priority of relatedness among person and setting and activity" (p. 180). Hence, the contextual features of the practicum classroom that influence preservice teachers' MCK related decisions were included within the study design, to more fully understand how mathematical knowledge is put into practice within that context.

The second implication identified was the need for a methodological design to capture changes in mathematical knowledge as a product of classroom interactions. Rooney and

Schneider (2005) argue that as any knowledge is enacted, “it brings a focus to the contradictions and paradoxes associated with the social and individual application of knowledge” (Rooney & Schneider, 2005, p. 30). Through the practice of teaching, a situated activity where preservice teacher learning can take place (Lave, 1996), a preservice teacher refines, reorganises, or reconstructs their own mathematical knowledge as these contradictions and paradoxes are revealed and managed over the course of a mathematics lesson. Preservice teachers' MCK is therefore understood, in this study, to potentially evolve and transform during, and as a result of, professional experiences in the classroom, reflecting a view of knowledge that is continually shaped by one's experiences of the world.

To summarise the epistemological stance taken in this study, knowledge is viewed as dynamic, evolving, and context-dependent (Mason & Spence, 1999), shaped and reshaped by social experiences in the communities in which it is located (Bodner, 1986; Lave & Wenger, 1991; von Glasersfeld, 1984). This view of knowledge necessitated mathematical knowledge to be examined first, within the context of a mathematics classroom, and second, using techniques that probed the contextual circumstances that led to preservice teachers enacting knowledge.

3.2 Research design

This study required a methodological approach that aligned with the empirical stance taken of knowledge, yet was also sensitive to the complexity of studying knowledge-in-action. An interpretivist approach to research underpinned the overall design of the study. Interpretivist researchers view peoples' actions in the social world as inherently meaningful and study the meaning of those actions (Erickson, 1986; Mason, 1996; Schwandt, 2000). Interpretivist field work, according to Erickson (1986) requires the researcher to collect data from within the setting of interest so that the data can be interpreted with the context in mind, and meaning generated for peoples' behaviour. This is achieved by spending time within a particular setting and collecting data pertaining to peoples' actions and interactions within the setting (e.g., audio and video records, field notes, artefacts, etc.). In this study, the social setting of interest was the practicum classroom. Section 3.5.1 describes how the data were collected from within the classroom setting.

An interpretivist methodological approach was appropriate for this study because the participants' thoughts were considered as significant as their MCK related actions. Patton (2002) explains that interpretivist researchers are particularly interested in people's thoughts because they illuminate certain conditions under which an act might take place. In this study, the preservice teachers' reports of their MCK related teaching actions provide a fuller understanding of why preservice teachers enact particular MCK in particular circumstances. Interpretivist research considers the intentions of people within a particular context when interpreting their actions and interactions (Schwandt, 2000). Consistent with an interpretivist approach, the goal oriented decisions that preservice teachers make about MCK were investigated in this study.

Social action and interaction are understood by the interpretivist researcher in terms of the context in which the actions occur (Erickson, 1986; Patton, 2002; Schwandt, 2000). Similarly, this study sought to investigate the preservice teachers' choice of enacted MCK with respect to the practicum context and the live classroom context. By investigating both the visible aspects (teaching actions) and invisible aspects (decisions) of MCK enactment using an interpretivist approach, the researcher looked for (a) explanations regarding what was happening (actions and interactions) within a setting and (b) what those events might mean to the people involved in them (Erickson, Florio, & Buschman, 1980).

Data collected within a particular setting is reflected upon analytically by an interpretivist researcher using forms such as vignettes, interview quotes, and summary tables (Erickson, 1986). Those forms were used by the researcher in this inquiry to study the preservice teachers' decisions that lay behind their MCK related actions and in turn, to give greater meaning to those actions. Interpretivist research can include "quantifications of particular sorts" (Erickson, 1986, p. 119) and in this interpretive study, numerical summaries were used to reveal trends regarding enacted MCK and decision making, enhancing what is predominantly qualitative data overall.

Accessing the thoughts leading to preservice teachers' MCK related decisions is difficult because of the invisible nature of decision making. To better access the invisible thoughts of the participants that led to enacted MCK, aspects of introspective research methodology were chosen to compliment the methods associated with interpretivist research.

Introspective research methods informed the timing and design of the first data analysis phase and data collection techniques in the second phase of data collection. Introspection dates back to Aristotle and Plato (Boring, 1953) and is defined as "the examination or observation of one's own mental and emotional processes" (Pearsall & Hanks, 1998, p. 958). One can introspect on an event by verbalising his or her thoughts and feelings about a particular event. James (1890, in Boring, 1953) refers to the process of introspection as "looking into our own minds and reporting what we there discover" (p. 170). Introspective research methods are used to elicit data concerning thought processes from participants involved in an activity (Gass & Mackey, 2000). Gass and Mackey (2000) explain that proponents of introspection assume that people are able to access and verbalise their own thought processes.

One can introspect about an event as it unfolds or after it has taken place. Think-aloud procedures where a participant is asked to verbalise their thought processes as they perform a task (Ericsson & Simon, 1993) can provide access to introspective thought processes (Lyle, 2003) as an event occurs. Alternatively, stimulated recall, a subset of introspective methods (Gass & Mackey, 2000), can be used to generate data after an event has ended. Stimulated recall procedures invite participants to recall their thoughts about an event, retrospectively, and are a valuable tool, according to Lyle (2003) to examine cognitive processes, including decisions. This study utilised stimulated recall procedures in the post-lesson interviews with the participants as a means to elicit data regarding their MCK related decisions. This introspective method allowed the participants the opportunity to look at their own MCK related teaching actions, and to verbalise their associated thought processes. Details of the stimulated recall interview are provided later in this chapter.

Introspection studies were discredited at the beginning of the twentieth century when Behaviourism became popular, due to the unreliability of results in psychological studies (Ericsson & Simon, 1993). However, interest in inner experience and the use of introspection-like questions and self-reports has continued to increase despite significant validity concerns (Hurlburt & Heavey, 2006; Vesterinen, Toom, & Patrikainen, 2010). This apparent contradiction highlights both the strengths and weaknesses of introspective methods. The errors which the human memory is inclined to make, and the lack of independent corroborating evidence available, creates substantial problems with reliability and researchers are encouraged to view any self-report with some scepticism. Nevertheless, the introspector has private access to any event that he or she recalls and given the uniqueness of this perspective, researchers are seeking to improve the use of introspective techniques in order to increase the accuracy and validity of reports and produce interesting and beneficial findings (Hurlburt & Heavey, 2006; Lankshear & Knobel, 2004). Several guidelines are suggested in introspection literature to increase the accuracy with introspective techniques and the researcher's adherence to these guidelines is more fully explained later in this chapter.

3.3 Study participants

The participants in this study were secondary mathematics preservice teachers in either their third or fourth year of study in a four year Bachelor of Education degree program at a regional Queensland university. The choice of university was an opportunistic one, as the researcher was a sessional lecturer of several teacher education courses at this university prior to and during the study. The participants deemed most likely to provide rich data about MCK were preservice teachers engaged in their third or fourth year of study, with the following characteristics:

- Successful completion of at least four first and second year tertiary advanced mathematics courses;
- Successful completion of at least six general education courses;
- Successful completion of the mathematics education course for lower secondary mathematics preservice teachers;
- Successful completion of at least first and second year practicums (three practicum phases);
- Enrolment in the professional experience (practicum) course for the current year, teaching at least one lower secondary mathematics class.

Potential participants, according to these characteristics, had arguably experienced some degree of success enacting MCK inside and outside the classroom context. Competence in procedures relating to advanced mathematics topics, such as matrix algebra, calculus, and analytic geometry was expected of the participants who had successfully negotiated their way through four tertiary advanced mathematics courses. Successful completion of the mathematics education course for lower secondary preservice teachers provided evidence that potential participants had some experience teaching mathematical content for a small number of secondary mathematics topics. The course included two mandatory teaching demonstrations to peers in a scenario-based teaching and learning context and eight one-on-one tutoring sessions with a lower secondary school student. Developing competency in teaching lessons in live classrooms had also been demonstrated to a degree because the preservice teachers had successfully completed at least three phases of professional experience in their prior years of study. General education courses were a significant part of the education program at the university where the study took place but did not feature regular, specific references to mathematics students, mathematics content, or pedagogical approaches for secondary mathematics.

Preservice teachers with the five characteristics were invited to participate. Over the two years of data collection in 2012 and 2013, ten eligible preservice teachers were invited to participate and all consented to do so. Data from six preservice teachers were collected in the first year of data collection and four preservice teachers were involved in the second year. After the data were collected, a decision was made to analyse data pertaining only to lower secondary algebra lessons (see section 3.5.3 for details). This resulted in the removal of four participants from the study, leaving six preservice teachers whose lesson data were analysed. The six participants comprised four men and two women, ranging from 20 to 23 years of age. All preservice teachers were excited to have been "chosen" for the study and willingly complied with all of the researcher's requests during the data collection phase. Details regarding the nature of their mathematics education program, their prior mathematics and mathematics teaching experiences, and the practicum context in which the observed lessons were situated are provided in the next section.

3.4 Context of the study

This study took place in Queensland at a time when professional teaching standards were being reviewed nationally. Queensland was the first Australian state to mandate teacher registration and the Queensland government has maintained a statutory body to oversee professional teaching standards for the last 40 years (McMeniman, 2004). At the time of the study, the study participants were completing a Bachelor of Education program that was accredited by the Queensland College of Teachers (QCT), the statutory body at the time. The four year program was designed to develop preservice teachers' competency in ten professional teaching standards (see Appendix B) outlined by the QCT (2006). Notably, no QCT standard explicitly referred to teachers' content knowledge.

Since the collection of the data for this study in 2012 and 2013, Australia has implemented a national set of teaching standards. The QCT teaching standards have been replaced with seven national standards (refer to Appendix C), developed by the Australian Institute for Teaching and School Leadership (AITSL). The *National Professional Standards for Teachers* (AITSL, 2014) make explicit reference to the importance of teacher knowledge in the second professional standard, "Know the content and how to teach it". A pertinent subsection of the second professional standard, specifically for those teachers in the early stages of their teaching career, states that graduate teachers "demonstrate knowledge and understanding of the concepts, substance, and structure of the content…of the teaching area" (AITSL, 2014).

The explicit attention paid to graduate mathematics teachers' content knowledge in Australia has increased in recent years. Five multi-institutional Australian research projects aimed at improving secondary mathematics and science education programs are currently in progress, supported by Australian Government (*Enhancing the Training of Mathematics and Science Teachers Programme)* funding (Office for Learning and Teaching, 2013). Although teachers' content knowledge is now more firmly on the Australian teacher education agenda, it was not strongly emphasised in the preceding QCT standards in Queensland or consequently, in the teacher education program undertaken by the participants of this study. Nevertheless, the QCT did have requirements concerning the secondary mathematics teacher education programs, discussed further in section 3.4.2.

3.4.1 The teacher education program

Entry into a secondary mathematics teacher education program in Queensland requires a passing grade in English and any one of three different mathematics subjects. A snapshot of the study participants' secondary mathematics achievements, all undertaken in Queensland secondary schools, is provided below in Table 5. Year 12 is the final year of secondary school in Australia.

Table 5. Summary of participants' secondary mathematics results

Preservice teacher (Pseudonym)	**Mathematics result at the end of Year 10**	**Mathematics result at the end of Year 12***
William	A	Ma B: A Ma C: B
Sam	A	Ma B: A Ma C: B
Thomas	A	Ma B: A
Kate	A	Ma B: A
Grace	A	Ma B: A
Ben	C	Ma B: B

** Ma B refers to Mathematics B, a senior secondary mathematics subject involving the study of functions, differentiation, and integration. Ma C refers to Mathematics C, a more advanced secondary mathematics subject, involving the study of matrices, vectors, and complex numbers. An 'A' indicates a very high level of achievement, the highest level of achievement awarded, a 'B' indicates a high level of achievement, and a 'C' indicates a satisfactory level of achievement.*

Table 5 shows that all participants, with the exception of Ben, were consistently high achieving mathematics students in secondary school. Ben commented in his interview that he attended a mathematics workshop (not at his own school) at the end of Year 10 and attributed his improved understanding of mathematics and his improved senior mathematics results to his experiences in the workshop.

The study participants' undergraduate program comprised university-based and school-based learning experiences at a regional Queensland university. In 2012 and 2013, when the data for this study were generated, the number of secondary mathematics teaching graduates was four and ten respectively. The low number of graduates reflects the small size of the secondary mathematics teaching cohort at this regional university.

In the university setting, participants were required to complete a minimum of 12 general education courses, four undergraduate mathematics courses, and only one mathematics education course to meet the requirements for a qualification to teach lower secondary (Years 8-10) mathematics. For an upper secondary mathematics education qualification (Years 8-12), four additional undergraduate mathematics discipline courses and one additional mathematics education course were required. Two of the participants in this study were studying to teach only lower secondary mathematics (Years 8-10), while the other four participants were seeking a qualification to teach both lower and upper secondary mathematics (Years 8-12). Table 6 provides details of the participants' progress in their secondary mathematics education degree at the time that data pertaining to their lessons (either in 2012 or 2013) were collected.

Table 6. Participants' backgrounds in the initial teacher education program

Preservice teacher	**Year of the degree**	**Teaching areas (Year 12/ Year 10)**	**Completed courses**		
			Undergraduate mathematics	**General education**	**Mathematics education**
Grace	4	Mathematics/ Drama	8	12	2
Ben	4	Mathematics/ Physics*	7	12	2
Thomas	4	Physical Education/ Mathematics	4	8	1
Kate	3	Mathematics/ English	6	7	1
William	3	Mathematics/ Technologies	5	8	1
Sam	3	Music/ Mathematics	4	7	1

** Ben undertook additional tertiary courses to obtain a qualification to teach both mathematics and science (Physics) to Year 12.*

Table 6 shows that the participants in this study were in their third and fourth (final) years of study. All participants had completed a minimum of four advanced mathematics courses, seven general education courses, and one mathematics education course.

In the school setting, participants were required to successfully complete 75 days of professional experience within eight practicum phases, across their four year undergraduate program. However, there was no expectation that the preservice teachers studying to become only lower secondary mathematics teachers (Sam and Thomas) would teach any mathematics lessons before their fourth year of study. The participants' opportunities to teach mathematics during their past practicums are presented in Table 7 for each practicum year. An approximate total number of mathematics lessons that each preservice teacher could remember teaching across their previous practicums is also shown in the table for each participant.

Table 7. Participants' opportunities to teach mathematics during their past practicums

Practicum phases	Total minimum teaching required (in both teaching areas) during phase	Mathematics taught in each practicum year					
		Grace	Ben	Thomas	Kate	William	Sam
Year 1 Phase 1 (1 week, April)	2 lessons	Yes	No	No	No	Yes	Yes
Year 2 Phase 1 (1 day/fortnight for 10 weeks, March-May) Phase 2 (2 weeks, July)	4 lessons 12 lessons in one teaching area	Yes	No	Yes	Yes	Yes	Yes
Year 3 Phase 1 (1 week, April) Phase 2 (3 weeks, July)	2 lessons in first (Year 12) teaching area ½ load in first two weeks, full load in final week	Yes	Yes	No	Yes	Yes	Yes
Year 4 Phase 1 (1 week, February) Phase 2 (2 weeks, April) Phase 3 (4 weeks, July/Aug)	1 lesson in each teaching area ½ load in first week, full load in second week Full load for 3 continuous weeks	Yes	Yes	Yes	n/a	n/a	n/a
Total number of mathematics lessons (approximate)		Over 30	Over 20	9	Over 30	12	9

Table 7 shows that the participants' experience in teaching mathematics varied from less than ten lessons over two years to more than 30 lessons over four years. The participants' experiences in teaching algebra showed less variation. No participant could remember teaching algebra prior to their current practicum year. Instead they taught topics including measurement, probability, finance, trigonometry, and geometry. William and Grace did teach one algebra topic (linear/non-linear relationships and binomial expansion, respectively) in a practicum phase earlier in the same year but did not reteach the same topic in the observed lessons. Overall, the preservice teachers had no classroom experience teaching the algebra topics that they delivered in the lessons observed in this study.

3.4.2 The practicum

The preservice teachers were completing their practicums in state government and catholic secondary schools located across the North Queensland region. Preservice teachers at the participants' university are encouraged to undertake out-of-town practicum placements during their four year program. Each participant was allocated one supervising teacher for the mathematics teaching component of their practicum.

In Australia, school-based educators who supervise preservice teachers are not always qualified mathematics teachers. More secondary mathematics teachers with robust mathematical knowledge are needed in Australia (Brown, 2009; Harris & Jensz, 2006; Hughes & Rubenstien, 2006; McPhan, Morony, Pegg, Cooksey, & Lynch, 2008), due to a shortage of trained secondary mathematics teachers in Australian schools (Ingvarson, Beavis, Bishop, Peck, & Elsworth, 2004; Lawrance & Palmer, 2003; Sullivan, 2011; Thomas, 2001). It is estimated in Australia that out-of-field mathematics teachers teach approximately 40% or more of lower secondary mathematics classes (McKenzie, Rowley, Weldon, & Murphy, 2011). For the six participants of this study, two were supervised by out-of-field mathematics teachers (Kate and Thomas), two were supervised by qualified secondary mathematics teachers (Ben and Sam), and the remaining two were supervised by mathematics heads of department (Grace and William).

The focus of the practicum mentorship may not be on content knowledge but other, more generic aspects of teaching. At the time of this study, the supervising teachers were not required to comment on the preservice teachers' content knowledge in their practicum reports because the criteria were aligned with the QCT standards which did not explicitly refer to content knowledge. Therefore, a lack of explicit attention may have been paid to the preservice teachers developing the MCK that is essential to the work of teaching, particularly for the two participants who were mentored by out-of-field mathematics teachers.

All preservice teachers stated that their supervising teachers had put no limitations on their lessons regarding their pedagogical approach. For example, Sam remarked, "As long as I relate it back to, obviously, what they have to be getting through, then she [the supervising teacher] doesn't usually have a problem with anything." In this study, all preservice teachers appeared to stay in a pedagogical comfort zone for all or most of their lessons. They regularly undertook teaching actions considered by Silver and Smith (1996) as very traditional ones for a mathematics classroom, such as transmitting knowledge and validating answers. No preservice teacher chose to use small-group work or co-operative learning (Killen, 2013) as a teaching strategy. The majority of employed strategies belonged to one or more aspects of the direct instruction teaching strategy (Killen, 2013; Lasley, Matczynski, & Rowley, 2002). These aspects are provided in Appendix D and show only two very brief deviations (less than five minutes) from the direct instruction path, taken by two participants.

The behaviour of the classes observed was of a very high standard. The majority of students appeared to listen attentively to the preservice teachers' explanations and contribute pertinent comments and questions when invited. No major behaviour management issues were noted across the lessons observed in the study. Although behaviour management can affect preservice teachers' instructional decisions (Westerman, 1991), in this study, the absence of any references by the participants to student behaviour may be explained by the particularly high level of behaviour exhibited by the student cohorts.

3.5 Data collection

Qualitative data is rich, by the very nature of the detailed descriptions it provides and is also situated or sensitive to the contexts in which the data are collected (Best & Kahn, 2006; Cohen, Manion, & Morrison, 2000). Hence, the complex phenomenon of live mathematics teaching for the preservice teacher required data collection methods to capture rich, situated data of their experience. Data collected from a combination of sources also allowed for "cross-data validity checks" (Patton, 2002, p. 248) and contributed towards a more accurate and less biased interpretation of the data (Lichtman, 2010).

The data collection approach devised for this study comprised two phases. Written lesson observations, video recordings, and the collection of teaching artefacts and contextual information took place in the first phase of data collection. Individual interviews featuring stimulated recall techniques comprised the second phase of data collection. Each phase is fully described below.

3.5.1 First phase of data collection

The data that were first collected from each participant were located within the context of most interest in this study; the live classroom. Each preservice teacher who originally consented to participate in the study (n = 10) was observed teaching two lower secondary mathematics lessons. The setting of a live classroom for observations, rather than more contrived scenario settings beyond the practicum context was chosen so that the observations conducted reflected the reality of the experiences of the preservice teachers more accurately (Burns, 1996; Best & Kahn, 2006; Erlandson, Harris, Skipper, & Allen, 1993; Lincoln & Guba, 1985). The use of a more artificial setting would have jeopardised the internal validity of any data collected (Burns, 1996; Patton, 2002).

The collection of data was a slow process, hindered primarily by the varied locations of the schools and the restrictions of practicum phases and secondary school teaching timetables. Accessing live mathematics lessons involved a great deal of forward planning but with flexibility embedded in the plans to accommodate last minute changes. The lessons were conducted across seven secondary schools, in three North Queensland towns and cities, up to 360km apart. After ethics approval from the university was received to approach each participant and school, and the permission of participants, school principals, heads of departments, supervising teachers, and for some schools, parents, was sought and received, scheduling of the researcher's visits to each school was arranged. In some cases, preservice teachers knew weeks in advance of their lesson times but for others, they were only able to give the researcher a couple of days' notice. Given the physical distance between schools, the short notice for the researcher to organise observation times and follow up interviews, and the limited number of practicum weeks when data could be collected, the researcher was limited in the amount of data that she could collect during practicum phases. In one week, it became feasible for the researcher to collect data relating to up to four lessons, from two preservice teachers, based at up to two different schools across the North Queensland region.

Participants and the researcher negotiated lesson observation dates and times via emails, text messages, and phone calls. The researcher was proactive in ensuring that the participants' behaviour was as authentic as possible, as recommended by Burns (1996). The researcher repeatedly stated to the participants in emails and phone calls prior to the observations and in conversations on the day of the observations that the participants should plan no differently to the way they usually would nor to teach their lesson any differently. The researcher was aware from comments made by some participants leading up to the observations that they were looking to impress the researcher, one of their former university lecturers, with their lessons. The preservice teachers' comments implied that they may have been preparing to modify the way they usually planned and implemented the observed lessons, so several additional reminders and clarifications regarding the researcher's expectations of the preservice teachers were given to these participants. Despite the attempts of the researcher to observe lessons in their most natural setting, it is possible that some lessons may have had extra attention paid to them as a result of having the observation take place.

The researcher collected observation field notes, video footage, and lesson artefacts pertaining to 20 practicum lessons. Ten were selected for analysis in this study. Section 3.5.3 outlines the researcher's decision to reduce the lesson data analysed to the lessons which focused on algebra.

3.5.1.1 Lesson observation field notes

Observation is a highly valued method of data collection because it allows the researcher to gather data from live situations (Patton, 2002). Observation can produce a better understanding of contexts and complexities of events, allow for the discovery of ideas that participants may not talk about in interview situations, and provide access to personal knowledge in action (Cohen et al., 2000; Patton, 2002). Perhaps the greatest benefit of observation is that it makes it possible to record behaviour as it occurs, with the implicit assumption behind observation that behaviour is purposive (Burns, 1996). Given these strengths, observing preservice teachers teaching in real classrooms was included in the first phase of data collection in the study. The recording of observation data via field notes provided the researcher with a collection of preservice teacher actions that potentially evidenced MCK being enacted.

Preparations were undertaken prior to the lesson observations to narrow the focus of what was being observed and recorded in the field notes. In early stages of qualitative research, researchers can typically approach an observation setting without predetermined categories (Marshall & Rossman, 2006). Borich (2008), however, notes the complexity of teaching practice and advises observers to choose a particular lens through which to observe the events in a classroom. Keeping in mind that all aspects of teaching practice could not be captured adequately, the researcher used focused observation, where the observer looks only at material that is pertinent to the issue at hand (Angrosino, 2005). Predetermined categories were established prior to the observations taking place with the purpose of capturing classroom events that related specifically to preservice teachers enacting MCK. The researcher designed a field note template (Appendix E), as recommended by Gay, Mills, and Airasian (2009), to guide and structure the lesson observations. It reflects the omission of certain aspects of the classroom experience, with a clear focus upon aspects of the lesson that were judged to more directly relate to how the teacher might mediate the mathematical content of the lesson.

A second field note template was designed to record the contextual features of the live lessons. A detailed description of a phenomenon, according to Lichtman (2010), requires contextual details about the setting of the study and those being studied. Accordingly, the researcher recorded as part of the observation field notes, details about the physical arrangement of each classroom, the perceived mood, behaviour, and attitude of the students and teachers, and the general class dynamic as a whole. A copy of the template, also completed by the researcher during the lesson can be found in Appendix F.

Three potential obstacles regarding lesson observation were noted for this study. Those obstacles comprised observer bias, complex patterns of behaviour which can confuse the observer, and recall problems if the recording of the observations is not immediate (Kellehear, 1993). The researcher recognised that capturing all MCK related actions as they occurred using only field notes was unrealistic and that field notes could only capture those actions that caught the researchers' attention during the lesson or were remembered to some degree by the researcher after the lesson. In an attempt to minimise those issues and to prepare stimulus for the second phase of data collection following each observation, all lessons were digitally videorecorded.

3.5.1.2 Video footage of the lessons

Digital recording of all observed lessons took place to capture the participants' teaching actions. The use of film to record data had two major advantages for this study, which are supported by Kellehear (1993). Firstly, the video footage provided a stimulus for the individual interviews that were undertaken in the second phase of data collection, detailed in section 3.5.2.1. Secondly, the footage proved to be a valuable resource for the researcher during the first and second analysis stages of the study. The audio records of the verbal interactions that took place provided the researcher with the opportunity to analyse the preservice teachers' mathematical explanations, including their use of mathematical terminology. The footage also offered the researcher the opportunity to review the data, discovering previously unnoticed subtleties in the context surrounding certain preservice teachers' actions. The researcher was able to reflect upon MCK related actions with the hindsight of having seen how the lesson unfolded. The benefits provided by the video footage in this study echo Best and Kahn's (2006) argument for a video record of observations to be used to interpret meaning behind observed behaviours.

The researcher's objective when recording classroom observations was to capture data that were as authentic as possible, recognising that the very act of recording will interfere with regular classroom dynamics (Lankshear & Knobel, 2004). A video camera was placed as unobtrusively as possible along the back wall of each classroom. The researcher stood or sat next to the video camera for the duration of the lesson and did not move around the room during the observation period. The video camera view generally followed the preservice teachers' movements around the room. In certain situations, the preservice teacher was standing away from a whiteboard or projected screen image but was referring to what was written on the board or screen. This meant that the preservice teacher's face could not be captured in the footage without compromising the capture of the visuals to which he or she was referring. In these circumstances, a decision was made by the researcher to give priority to the visuals appearing on the screen (whiteboard or projector) over the facial expressions and body language of the preservice teacher. The researcher believed that more pertinent information regarding a preservice teacher's MCK could be drawn from verbal descriptions offered in conjunction with mathematics content provided visually on the screen.

3.5.1.3 Teaching artefacts

Artefacts are important sources of data, which Erlandson et al. (1993) describe as further strands of colour and shape which can be added to a rich tapestry of data collection methods. As artefacts can contribute to a better understanding of what is happening in any practice (Lave & Wenger, 1991), the researcher used the data provided in the artefacts to better understand how the MCK of the preservice teachers came to be enacted. Participants provided the researcher with electronic or hard copies of teaching artefacts related to each lesson observed. Those artefacts included lesson plans, school term overviews, textbook pages, and lesson worksheets. It was anticipated that the level of detail in the lesson plans would vary from one preservice teacher to the next, as the researcher repeatedly advised each of the preservice teachers not to plan any differently from how they usually would plan their practicum lessons. This resulted in some very sparse lesson plans from some of the participants. Nevertheless, the lack of detail in written planning was considered preferable to detailed but contrived lesson plans that might have distorted the picture of how the preservice teachers usually prepared to deliver mathematical content in a lesson.

Observation field notes, video footage, and lesson artefacts pertaining to each observed lesson provided the researcher with visible manifestations of the preservice teachers' MCK. The researcher chose not to supplement the data collected from the study participants with data from the supervising teachers because the focus of the study was on the preservice teacher experience of enacting MCK. To access the less visible aspects of the preservice teachers' enactment of MCK, data collected in the first phase of data collection were used to inform the collection of data in the second phase.

3.5.2 Second phase of data collection

The preservice teachers' interview responses and background information comprised the data collected in phase two. Together, they contribute to the study in three ways. First, the data were used to contextualise the lessons observed, providing valuable information about the preservice teachers' perceived influence of a number of contextual factors including, but not limited to, the supervising teacher, school documents, and class textbooks. Second, the data gathered provided insights into the participants' thoughts that lay behind their MCK related actions, including their pedagogical intents, classroom circumstances that caught their attention, and knowledge they drew on as they made MCK related decisions. Third, the data provided the researcher with an avenue through which to verify the MCK that preservice teachers enacted in their lessons and to unearth MCK that informed teaching decisions but was not directly observable in the lesson.

To gather the information described, semi-structured, stimulated recall interviews were undertaken. The interview responses provided the majority of the data in phase two but were supplemented with informal discussions with the preservice teachers regarding their mathematics background either before or after the interviews. A full description of the interview process and informal conversations is provided, including a justification for their inclusion in the study.

3.5.2.1 The stimulated recall interview

Subsequent to the lesson observations, interviews were undertaken to gain a better understanding of the lesson from the preservice teachers' perspective. The interview has distinctive advantages as a data-gathering technique (Best & Kahn, 2006) because it can provide information that cannot be accessed through observation alone (Gay et al., 2009). Interviewees are given the opportunity to express how they interpret a situation from their own point of view (Cohen et al., 2000). As teachers are in the best position to gain access via introspection to their own intentions in a teaching situation (Burns, 1996), the researcher had the unique opportunity to enter the preservice teachers' world and to "walk a mile" (Patton, 2002, p. 416) with each participant. The pairing of interviews and observations was therefore a valuable way to gather complementary data and the use of multiple methods further strengthened the study by triangulating data about the same phenomenon from different data sources (Gay et al., 2009; Guion, Diehl, & McDonald, 2011; Patton, 2002).

Interviews have been used successfully in studies investigating preservice teachers' MCK, providing an opportunity for researchers to clarify and deepen their understanding of the participants' MCK. Interviews with introspective elements have been effectively employed to more fully understand how MCK has been enacted in mathematics questionnaire responses involving secondary preservice teachers' understanding of division (Ball, 1990) and in preservice teacher actions and discourse when teaching secondary mathematics lessons (Rowland et al., 2011). Ball (1990) for example, designed interview questions featuring introspective techniques to probe mathematics preservice teachers' knowledge and beliefs concerning mathematics, mathematics teaching and learning, and mathematics students. The questions successfully elicited data showing that preservice teachers could perform procedures involving division of fractions but that they lacked strong conceptual understandings of the procedures. Data concerning preservice teachers' beliefs about the nature of mathematics (as predominantly rules and facts) and student ability (innate or a product of effort and desire) were also elicited from introspective techniques.

Rowland et al. (2011) used introspective techniques to gain access to a preservice teacher's reflections of their MCK related teaching actions. The researchers conducted a stimulated recall interview with the participant, showing her lesson excerpts to prompt reflections on the participant's thinking at the time. The preservice teacher's recall of her thoughts as she taught a mathematics lesson revealed constructivist beliefs about teaching and learning mathematics. The preservice teacher's intentions for performing particular teaching actions were also expressed in the stimulated recall interview. The stimulated recall interview is therefore a valuable data collection tool from which to collect information relating to preservice teacher MCK.

This study required specific information regarding MCK and decision making to be elicited, so a carefully designed structure was needed to maintain the content validity of the data collected (Best & Kahn, 2006). A semi-structured interview type (Lankshear & Knobel, 2004; Punch, 2009) was chosen as the most appropriate way to obtain data pertaining to particular topics (e.g., the participants' perceived comfort level teaching the lesson content) whilst still providing opportunities for participants to be probed for further responses and for unexpected insights to be discovered (O'Toole & Beckett, 2010). The general structure of all interviews followed the protocol detailed in Appendix G, which was modelled on one used successfully by Ethel and McMeniman (2000). What follows is a summary of the interview structure:

1. Brief discussion regarding the preservice teachers' overall impression of how the lesson progressed, from a mathematical content point of view;

2. Brief discussion of how the lesson was planned, including the preservice teachers' mathematical preparations and their perceived levels of autonomy in planning the lesson content that they delivered;
3. Extended discussion using stimulated recall procedure (most of the interview): Preservice teachers watched edited footage of the lesson and provided commentary of their thoughts during the lesson;
4. Brief discussion of the preservice teacher's perception of the lesson after having reflected on some aspects of the lesson.

Interviews were initially planned to take between 30 and 40 minutes but they often extended well beyond the agreed time either with enthusiastic agreement or sometimes at the request of the participants. Of the six participants who taught an algebra lesson, the first interview completed with each preservice teacher was, on average, 48 minutes in length. For those who participated in a second interview about an algebra lesson, the average interview time was 31 minutes because the explanations required for point 2 in the interview structure just described, were confirmed by the preservice teachers as being the same as the first lesson. The preservice teachers chose to be interviewed on the school grounds of their practicum school and in each case, an empty office or classroom was used for the interview. Video recording of the interviews was completed, providing a convenient and unobtrusive way to gather data and removing the necessity of the researcher having to take notes during the interview, which the participants may have found distracting (Best & Kahn, 2006). The video camera was pointed at the researcher's laptop screen showing the edited video footage and not at the participant. This allowed the researcher to align the participants' commentary with the on-screen teaching actions and to capture any instances when the participants gestured towards certain parts of the screen during the interview.

Stimulated recall was chosen as the most appropriate technique to elicit preservice teachers' thoughts about their MCK related teaching actions. The stimulated recall interview is a research procedure where video excerpts are played to participants, inviting them to recall a particular event and their thinking at that time (Lyle, 2003). In this instance, the events played back to each participant were video excerpts of the preservice teacher's own lesson which the researcher had selected because these evidenced the preservice teacher putting his/her MCK into practice. The process used to select the excerpts is described later in this chapter. A key aspect of the stimulated recall process is inviting participants to articulate their own thinking and it is this aspect which is characteristic of introspective techniques.

There were a number of advantages to the use of the stimulated recall technique for the researcher. In order to be as unobtrusive as possible when observing the lessons, which is considered essential due to the naturalistic setting of the observation (Lyle, 2003), the researcher was unable to confirm or clarify any inferences about the preservice teachers' MCK related actions during the observation. Having the stimulated recall interview available allowed the researcher to collect classroom data without disrupting the lesson, knowing that an opportunity to further clarify and explore the researcher's inferences about how the preservice teacher enacted MCK would follow at a later date.

The reports by the preservice teachers of their own thinking at key junctures during the observed lessons served to enhance the researcher's interpretations of their in-class actions which is another advantage of stimulated recall interviews. The preservice teachers' contributions allowed the researcher to more accurately describe and more confidently interpret the events in the lessons by hearing the participants' own interpretations of the same events. It is perhaps for this reason that stimulated recall methods are considered the "least intrusive but most inclusive way of studying classroom phenomena" (Reitano & Sim, 2010, p. 218).

Introspection, as alluded to earlier in this chapter, requires certain conditions to be met if recalled data from a participant is to be considered reliable. One condition is the necessity to target precise, concrete episodes in order to access specific episodic memories (Hurlburt & Heavey, 2006), as explained earlier, and the use of video footage allowed the researcher to meet this condition. Video footage provides a vivid and accurate reminder of events, can improve the coherence of the participants' thinking, and helps them to gain an insight into their own cognitions, including their own knowledge (Bloom, 1954; Ericsson & Simon, 1993; Meijer, Zanting, & Verloop, 2002; Reitano & Sim, 2010).

At the beginning of each interview, the researcher explained the process of stimulated recall to the participants (see Appendix G). The researcher encouraged the participants to first, pause the footage at any time that it stimulated a memory regarding their thinking at the time and second, describe to the researcher what they were thinking about. The researcher also added that if there was a particular event that the preservice teacher did not discuss, the researcher herself may pause the footage and ask a question about the event. Although the preservice teachers were understandably nervous about watching themselves teach, they appeared fascinated by the outsider's view provided by the video camera and were generally very animated in their reflections. In most instances, the participants paused the video footage regularly and spoke at length about their thinking at the time. There were occasions, however, when the preservice teachers watched the footage intently, but did not pause the footage to speak about it. In these instances, the researcher paused the video at stages of interest to her and used prompting statements and questions to stimulate a response from the participant.

The questions asked by the researcher, prepared before the interview in case they were needed, varied according to the degree of detail included in the questions themselves about the on-screen events. At first, as recommended by Gass and Mackey (2000), the questions were non-directive, such as, "What was happening here?", or, "What were you thinking at this point?" If the researcher sensed that further probing or highlighting of an event might provide insight into a situation, participants were prompted with statements of what could be seen on the screen, such as, "You looked surprised by that response," or, "I noticed that you erased the first example, replacing it with this second example." Appendix G contains a more detailed list of prompts that were used in the interviews, adapted from Powell's (2005) work. The researcher used as few directed prompts as possible, following Meade and McMeniman's (1992) advice to avoid "lead[ing] the witness down the path of the researcher's thoughts rather than those of the teacher" (p. 3).

The timing of interviews was planned judiciously, given the introspective aspects of the interview. Introspective methods are most effective when they occur with as little delay as possible between the event and the recall of that event (Gass & Mackey, 2000; Hurlbert & Heavey, 2000; Lyle, 2003). Major sources of errors in social science research have been related to memory error, when missing pieces of memory are often filled in by the participant with their own general knowledge of a situation and this phenomenon increases as the delay between an event and its recall increases (Tourangeau, 2000). The time frame between the observation of the lesson and the stimulated recall interview was therefore kept to a minimum.

All interviews took place within two days of the associated observed lesson. The time needed for interview preparation by the researcher and the teaching commitments of the preservice teacher meant that an interview immediately following the observation was not possible. Although Rowland et al. (2011) gained valuable information regarding enacted MCK after a 20 day delay between observations and stimulated-recall interviews, the researcher chose to conduct each interview a maximum of two days after the class observation, keeping in mind the concerns about memory loss and following recommendations and methodological approaches of other studies utilising stimulated recall interviews (Bloom, 1954; Gass & Mackey, 2000; Kwon & Orrill, 2007; Wu & Badger, 2009).

3.5.2.2 Background of the preservice teachers

In qualitative research, Punch (2009) states that "we cannot give the full picture unless we have the full picture" (p. 161) indicating a need for the researcher to collect sufficient information about the context in which an inquiry is carried out. While the "full" picture of preservice teachers enacting MCK in a live classroom could not be captured, the preservice teachers' experiences in mathematics and mathematics teaching were regarded by the researcher as potentially important considerations. Hence, background information about the preservice teachers' prior mathematics learning and teaching experiences was collected for all participants and reported earlier in this chapter. The participants' backgrounds also form part of the discussions presented in chapters 6 and 7. The information was gathered orally from all participants in conversations that took place either directly before or after one of their individual interviews. The participants' responses were not audio recorded but instead, the researcher took notes, using the template provided in Appendix H.

3.5.3 Reduction of lesson and participant data

Initially, data for twenty lessons were collected over two four week periods in July and August of 2012 and 2013. The periods of data collection were arranged to coincide with the practicum phases of the preservice teachers, some of whom were undertaking a three week placement (3^{rd} year students) and others whom were undertaking a four week placement (4^{th} year students). Observations were arranged to coincide with particular lessons, when preservice teachers knew ahead of time that they would be teaching a lower secondary mathematics class. At this stage, the focus was not to record only algebra lessons but instead, to capture whatever topics the preservice teachers were encountering in lower secondary mathematics. The researcher prioritised the authentic nature of a practicum lesson at the expense of a study design where lesson content could be dictated, but where lessons would need to occur outside of the practicum context.

After both data collection phases were completed in 2012 and 2013, a review of the 20 lessons was undertaken. Ten of the 20 lessons, taught by six preservice teachers, concerned the topic of algebra. The remaining ten lessons were to do with measurement (three lessons), statistics (two lessons), probability (one lesson), rates (one lesson), finance (one lesson), trigonometry (one lesson), and geometry (one lesson). The researcher also noticed in the algebra lessons that there were some remarkable similarities between the MCK of algebra that the preservice teachers enacted within the algebra lessons but also briefly in other lessons (e.g., calculations of areas and volumes using formulae). The researcher believed that analysis of the ten algebra lessons would produce findings that (a) could include trends across multiple algebra lessons and preservice teachers and (b) could potentially impact the quality of preservice teachers' lessons across other mathematical strands, as algebraic manipulations permeate the secondary school curriculum. Consequently, the researcher decided to reduce the lessons analysed to only those involving algebra and the data were reduced accordingly.

The final data set of this study comprised the data pertaining to ten lower secondary algebra lessons taught by six preservice teachers. Four of the preservice teachers were observed and interviewed regarding two sequential lessons occurring within the same week and the remaining two preservice teachers were observed and interviewed only once. The duration of the lessons ranged from 40 minutes to 100 minutes, with a mean lesson length of just over an hour. Six of the lessons were delivered to Year 8 classes and the four remaining lessons were presented to Year 10 classes. Although the lesson content of the Year 8 and Year 10 lessons differed, the type and quality of the MCK enacted was not significantly different because all of the lessons concerned solving equations.

The lessons observed took place either in the middle or towards the end of the algebra unit in which they were located. Table 8 identifies the location of each lesson or pair of sequential lessons in the overall unit for each preservice teacher and the mathematical topics covered in the observed lesson(s).

Table 8. Topic and location of observed lessons in algebra units

Preservice teacher	Observed lesson(s)	Year level	Content focus of the lesson(s)	Location in algebra unit	Content introduced prior to lesson(s)	Prior lessons taught by preservice teacher in current algebra unit
Kate	2	10	Solving simultaneous equations (substitution method and word problems)	Final week of a three week unit	Graphical, elimination, and substitution methods	6
Grace	2	10	Solving simultaneous equations (substitution method)	Final week of a four week unit	Graphical and elimination methods	7
William	1	8	Solving linear equations (balance method)	Final lesson of a one week unit	Manipulating algebraic expressions (expansion, simplification)	4
Sam	2	8	Classifying true/false equations; solving linear equations (word problems and backtracking method)	Middle lessons of a five week unit*	Manipulating algebraic expressions (expansion, simplification); classifying true/false equations; solving linear equations (transposing method)	4
Ben	1	8	Revising algebra unit content, including solving equations	Final lesson of a five week unit	Manipulating algebraic expressions (expansion, simplification); solving linear equations and quadratic equations of the form $ax^2 = c$ (balance method)	8
Thomas	2	8	Identifying algebraic terms and expressions; solving linear equations (word problems)	Final lessons of a four week unit	Manipulating algebraic expressions (expansion, simplification); solving linear equations (balance method)	8

** Sam's term overview did not indicate an overall length for the unit and because he was about to complete his practicum, Sam was unsure of how much longer the unit would be continuing. The researcher has approximated the unit length based on the content listed on the overview that was yet to be covered.*

Mathematics content that had been covered within the unit, but prior to the observed lessons, is also included in Table 8, to give an indication of the mathematical content that preservice teachers were expected to build upon in their observed lessons.

Following the reduction of the original data set to only those lessons concerning algebra, the lesson and interview data of the ten algebra lessons were prepared for analysis. An audio record of each interview, including the edited lesson footage that was played during the interview, was transcribed with the assistance of a transcription service. The researcher added screenshots of relevant lesson footage, including whiteboard examples or projected worked examples and exercises, and pertinent field notes to the interview transcript to provide a more detailed record of the lesson excerpts and the interview. This process produced one document for each lesson that incorporated details of the lesson and interview, in readiness for analysis.

3.6 Data analysis: Phases 1 and 2

This study reflects the cyclic nature of the qualitative research process, where data collection and analysis often proceed simultaneously (Boeije, 2010). Data analysis occurred during and after the periods of data collection, in three phases. The first phase of analysis occurred concurrently with data collection over the course of the study, between the first and second phases of data collection. The second phase occurred after all data collection phases were completed and the overall data set reduced to data concerning only the ten algebra lessons. The third and final data analysis phase is described in section 3.8 after the analysis framework has been introduced. In this section, the first and second phases of data analysis are elaborated.

3.6.1 First phase of data analysis

The first phase of data analysis took place during the data collection period and occurred after each class observation, in preparation for the imminent stimulated recall interview. Schepens, Aelterman, and Van Keer (2007) found in one study that a full lesson of footage was too onerous for preservice teachers to review given their teaching commitments. This phase therefore involved an analysis and reduction of the observation footage, teaching artefacts, and field notes in order to select the most valuable stimuli to present to the preservice teachers in the stimulated recall interviews. The results of the analysis were evidenced in the researcher's choice of stimulus, presented in approximately 15 minutes of footage to participants in their interviews.

Time was an important factor to consider in the design of the first phase of analysis. The analysis needed to occur in a relatively short period of time, in some instances between a morning lesson and an after-school interview. The method of analysis needed to be efficient, given the time restrictions involved, while also producing superior stimuli to which the participants would respond. Consequently, the lesson footage was reduced in three systematic stages to produce a set of stimuli that could feasibly be discussed in one interview.

The first stage in the reduction of the videotaped footage was to identify footage which the researcher considered less worthwhile and could therefore be excluded from the potential recall stimuli. The researcher was prepared to exclude observation footage in particular circumstances. Footage was excluded when it did not contain information relating directly to mathematics content. Issues of class routine and management of students were excluded, along with any class discussions that were not about the content of the lesson. Next, footage which duplicated a very similar mathematical situation occurring in the same lesson was reduced. When this occurred, the first excerpt was kept and later excerpts removed, allowing for more variety in the MCK evidenced in the stimulus footage. Later duplications were not excluded if a richer discussion took place or if the researcher thought that the MCK enacted by the preservice teacher differed in some way. Finally, footage was excluded during the independent practice stages of lessons if the preservice teacher moved around the room but did not stop to talk to students or the footage failed to audibly pick up quietly spoken one-on-one conversations with students.

After the initial exclusions of footage occurred, the researcher considered the length of footage remaining and if it was less than 15 minutes in duration, all remaining excerpts were kept. If the footage was longer than 15 minutes, as it was for most of the observed lessons, the researcher made further decisions regarding the footage to be included in the stimulated recall interview.

In the second stage of reducing the video footage, the researcher further prioritised video excerpts for use in stimulated recall interviews by selecting critical incidents. Extracting critical incidents as a methodological approach was developed in the early 1980s (Miles & Huberman, 1994) and sets of critical incidents have been used effectively to communicate the essence of a phenomenon in a school setting (Erlandson et al., 1993). Angelides (2001) found the use of critical incidents to be advantageous for generating rich qualitative data quickly and given the time constraints involved with this phase of analysis, the use of critical incidents was found to be both valuable and efficient.

The word 'critical' does not necessarily refer to an obvious or dramatic event but to any event where a significant feature of a phenomenon or an important insight into a person being studied is revealed (Cohen, Manion, & Morrison, 2000; Erlandson et al., 1993; Tripp, 2012). Often in direct contrast to remarkable events, critical events are frequently "straightforward accounts of very commonplace events that occur in routine professional practice which are critical in the rather different sense that they are indicative of underlying trends, motives, and structures" (Tripp, 2012, pp. 24-25). In this case, critical incidents were events where the preservice teachers were enacting particular aspects of their MCK that were of interest to the researcher. It can be argued that anything that happens in a classroom is a potential critical incident (Angelides, 2001) so it was the researcher's own interpretations about the lessons and participants which ultimately led to the final choice of critical incidents. Given the short time frame in which video footage was reviewed, it is possible that other equally valuable incidents may have been available to the researcher, but were not recognised as such.

The researcher first identified critical incidents during the lesson observation itself. Specific MCK related actions that caught the attention of the researcher were noted, no matter how ordinary or routine they appeared (refer to the "hot spots" column of Appendix E). The researcher relied upon the literature regarding preservice teachers' MCK to inform her choice of critical incidents. A number of scholars also refer to 'surprises' as stimuli for reflection by a researcher (Angelides, 2001; Schein, 1985; Schön, 1995). In this case, the researcher relied not only upon her research background but additionally her background as a secondary mathematics teacher with ten years of experience to identify what she considered were surprising or unexpected MCK related teaching actions. Reflection upon those surprising events from the researcher's point of view often led to the identification of more critical incidents.

The researcher, sensitive to the knowledge that critical incidents may present themselves in very subtle ways, watched the lesson unfold for a second time to identify more critical incidents. The researcher reviewed the video footage remaining after the first stage of footage reduction and noted any MCK related critical incidents that were not captured in the researcher's observation field notes. Having watched the whole lesson previously, the researcher was more able, in the second viewing, to identify critical incidents that showed each preservice teachers' "typical" enactment of MCK and any incidents where the preservice teacher enacted notably different MCK. In some cases, a more experienced tertiary mathematics educator also viewed the video footage and assisted in the identification of critical incidents to be explored in the interviews. The footage of all critical incidents was isolated onto a separate digital video file. The remaining footage, which was usually only a few minutes in length and did not feature significantly different enacted MCK to that of the critical incidents, was discarded. Once more, the duration of the footage that was kept was considered and if the duration of the footage was still more than 15 minutes in length, a final step was required to reduce the footage a final time.

The third and final stage of reducing the video footage, which was rarely required, involved omitting some of the critical incidents in favour of others. The strength of the stimulated recall interview is the introspective aspect, which allows the researcher, as an outsider, to access the data in the participant's head and it is this principle which shaped the final decision making process. Priority was given to those incidents where the researcher was unsure about how a situation had unfolded and was relying on the preservice teacher's perspective to clarify or explain an event. Priority was also given to those incidents where the researcher felt that an interaction might generate particularly rich data regarding MCK in the stimulated recall interview. Critical incidents that appeared less ambiguous were omitted when necessary, ensuring that the footage that remained could be fully explored within the time constraints of the interview.

On completion of the footage reduction process, the researcher was left with collections of lesson excerpts, comprising one or more critical incidents, extracted from across the duration of each lesson. The number of excerpts in each collection varied between three and 12, and ranged from less than 30 seconds to over ten minutes in length. However, in the majority of cases (seven of the ten lessons), the reduced footage comprised between four and six excerpts, each ranging between one and five minutes in length. The footage was prepared as a series of video clips capturing typical and atypical MCK related actions, arranged sequentially in the order they took place in the lesson, in a separate digital video file, ready for use in the interview.

3.6.2 Second phase of data analysis

3.6.2.1 Data reduction

The second stage of the analysis took place after all of the data were collected, collated, and transcribed. Like in the first phase, it involved data reduction, which Miles and Huberman (1994) indicate is a key process in qualitative analysis that "sharpens, sorts, focuses, discards, and organises data" (p. 11). Before the data could be organised or coded, it was necessary to reduce the data set to only those teaching actions where information was available to link the preservice teachers' actions pertaining to MCK with their thoughts that lay behind those actions. This was achieved by sorting through the interview transcripts and identifying all teaching actions that (a) the preservice teacher commented upon and (b) did so with respect to the content of the lesson and not other issues (e.g., behaviour management). Teaching actions relating to mathematics content that did not attract the attention of the preservice teachers, or where the preservice teachers indicated that they couldn't remember what they were thinking at the time, were discarded from further analysis. However, there were very few of these instances (one set of actions in four of the lessons). Most of the stimuli played to the preservice teachers in the interview attracted reflective comments and was retained for further examination.

The lesson and interview data were organised into units of analysis that featured enacted MCK and associated preservice teacher reflections. The researcher made several unsuccessful attempts to organise the reduced set of data into workable units of analysis, before finding a successful approach. The researcher tried partitioning the preservice teachers' MCK related actions into units of equal duration (10, 20, or 30 seconds) and also at a more micro level, into individual actions (e.g., a question, a statement, a written line of working, etc.). Neither approach was successful because the preservice teachers' reflections did not relate to the same units of analysis, varying instead from reflections on individual language or notation choices to reflections on a one minute conversation with a student. The researcher's intent was to create connections between preservice teachers' MCK related actions and decisions behind those actions. The approach needed to allow for those intended connections to be made. The researcher eventually found that the preservice teachers' lesson commentary could be used to partition the preservice teachers' MCK related actions into nested units of analysis, ensuring that MCK related thoughts behind related actions would be available for analysis in each unit.

3.6.2.2 Nested units of analysis

The reduced lesson footage, interview comments, lesson artefacts, and preservice teacher background information were organised into four nested tiers of data, shown in Figure 2. The researcher used QSR International's (2012) NVivo 10 qualitative data analysis software to sort the data into units of analysis and, when appropriate, to group smaller units of analysis within larger ones.

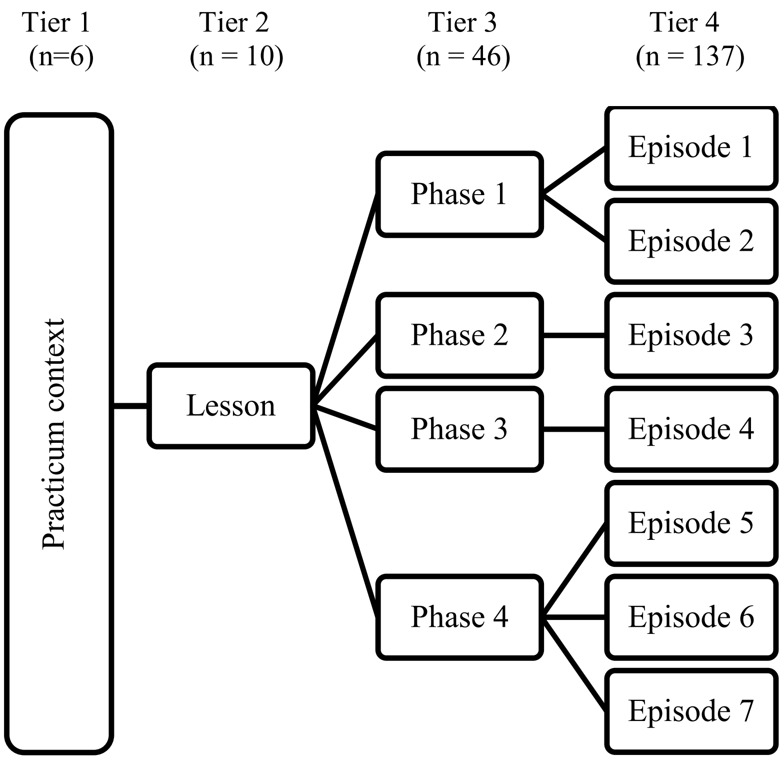


Figure 2. Nested tiers of data

Data pertaining to 137 small sets of teaching actions, known as episodes (tier 4) were identified and nested with data relating to 46 lesson phases (tier 3), which were, in turn, nested within the data to do with the ten overall lessons (tier 2). Finally, data for each lesson were situated with information relating to the six practicum contexts of the preservice teachers (tier 1). The researcher was able to discern elements that influenced the preservice teachers' MCK related decisions and subsequent actions from the data relating to each tier of data, supporting Zimmerlin and Nelson's (1999) finding that varying grain levels of analysis will result in a better understanding of teacher decision making than only fine grained analyses of single actions and decisions.

All data that related to the preservice teachers' overall practicum experience were organised into the first tier of data. Practicum data, sourced from interview transcripts and lesson artefacts, comprised (a) the preservice teachers' general comments regarding their practicum experience, (b) any reference they made in the interview to elements of the practicum context that influenced their MCK related actions in a general or specific way, (c) the researcher's field notes regarding the practicum context, and (d) school resources such as term overviews or class textbook pages. The second tier of data, also sourced from interview responses and lesson artefacts, related to the preservice teachers' impressions of, and planning for, the lesson as a whole. The process used to organise the lesson data into lesson phases and episodes, the third and fourth tiers of data, was less straightforward and is described below.

The researcher parsed the lesson footage and the preservice teachers' commentary on that footage to form the third and fourth tiers of data, using a two-step process. The process to decompose the lessons relied upon the preservice teachers' goals for different aspects of their lessons, discerned from their lesson commentary. Firstly, the lessons were divided into "lesson phases" (Artzt & Armour-Thomas, 1999, p. 214), which were segments or phases of lessons that were several minutes or more in length and defined by the pedagogical goals offered by the preservice teachers for those lesson phases. Secondly, each lesson phase was further deconstructed into "episodes", which comprised smaller sets of MCK related teaching actions. Just as the lesson phases were defined by pedagogical goals, so too were the episodes, which formed the smallest units of analysis in the study.

Lesson phases are an effective way to consider variations of teaching practice over the course of a lesson (Artzt & Armour-Thomas, 1999). In contrast to Artzt and Armour-Thomas (1999), who parsed lessons into three chronological lesson phases (i.e., initiation, development, and closure), the researcher identified and named the different types of phases according to the goals offered by the preservice teachers in their interviews. For example, most lessons included a "review content" lesson phase where the preservice teachers signalled their intention to review familiar content that students were expected to know. The four types of lesson phases identified by the researcher are described in section 3.6.2.3. Using the preservice teachers' goals to define the lesson phases provided an opportunity to compare and contrast the MCK enacted by the preservice teachers, according to broader pedagogical goals. Hence, the preservice teachers' references to lesson phases in their interviews comprised the data set of tier three.

After the lesson phases had been established, each lesson phase was decomposed into a number of episodes, comprising tier four of the data set. Once more, the preservice teachers' intents were used to indicate where one episode began and ended. An episode is a set of one or more sequential MCK related actions, observable in the lesson footage. Each episode is demarcated by one or more pedagogical goals for that set of teaching actions, either explicitly described or strongly implied by the preservice teacher in the interview. The researcher's notion of an episode is similar to Ribeiro, Monteiro, and Carrillo's (2009) and Clarke and Helme's (1997) use of the same term, which Clarke and Helme describe as a "coherent unit of activity unified by a single purpose" (p. 120). In Clarke and Helme's (1997) study, the purpose of a classroom episode was more closely associated with students, whereas in this study, the episode is unified by one or more teacher goals. Ribeiro et al. (2009), however, do use teacher goals to delineate episodes in the same way as in this study.

The duration of an episode varied according to the number of teaching actions performed in pursuit of the goal(s) underpinning the episode. An episode could take just a couple of seconds, if a preservice teacher made a specific verbal comment or wrote down a particular piece of mathematics content with a particular goal in mind. A longer episode could take minutes, comprising verbal questions, verbal comments, and written notation. However, that set of teaching actions need *all* be constrained by the same pedagogical goal(s) that the preservice teacher alluded to in their interview. In this way, the preservice teachers' pedagogical intention, an integral part of any teacher's decision making process (John, 2006; Leinhardt & Greeno, 1986; Schoenfeld, 2010; Shavelson & Stern, 1981; Simon, 2006; Westerman, 1991) informed the researcher's organisation of the preservice teachers' instructional practice into units of analysis, providing later opportunities to examine the connection between MCK related decisions and enacted MCK.

3.6.2.3 Coding process

Coding was undertaken after the four tiers of data were established. The coding process involved coding every tier of data for evidence of influencing elements impacting preservice teachers' MCK related decisions. Only tier four data (i.e., episode data) was coded for evidence of MCK enactment because this unit of analysis included the retained video footage of preservice teachers' MCK related actions (the episode) and the commentary on specific sets of actions (the episode reflection).

The practicum and lesson data were reviewed to identify elements that appeared to impact the preservice teachers' decisions to deliver particular MCK in all or part of a lesson. Evidence pertaining to the preservice teachers' MCK related decisions was coded using NVivo 10 software. Lesson phase data were also coded for influencing elements but as the data set was very small (preservice teachers only briefly mentioned or confirmed lesson phases), only lesson phase goals were discerned from the preservice teachers' comments. The goal types used to define, and subsequently code the lesson phases, were those that specifically referenced the level of familiarity that the preservice teachers expected their students to have with the content they presented. The four categories used to demarcate and code the lesson phases, generated first from researchers who had decomposed lessons into smaller parts (e.g., Artzt & Armour-Thomas, 1999; Australian Council for Educational Research, 1999; Schoenfeld, 2013) and refined after examining the preservice teachers' interview comments, are:

1. Introduce content: Intent to introduce mathematical content that students were not expected to have been exposed to before the current lesson (i.e., unfamiliar content). For example, William introduced his students for the first time to the process used to solve a linear equation.

2. Consolidate content: Intent to deliver mathematical content to which the students had already been introduced (i.e., familiar content), but that the preservice teacher did not expect they would have been able to work with independently. For example, Thomas knew that his students had been exposed to the words "term" and "coefficient" in previous lessons, but he provided a glossary of algebraic terminology to his students as part of his lesson because he believed his students did not fully understand the meaning of the words.

3. Develop content: Intent to advance students' knowledge of mathematical content with which they were already familiar, by exposing them to unfamiliar, but related content. For example, Grace knew that her students were familiar with the graphical and elimination methods of solving simultaneous linear equations when she presented the substitution method.

4. Review content: Intent to revise familiar mathematical content that the preservice teachers expected the majority of their students could work with independently. For example, Kate presented a series of warm up questions to review procedures that she expected her students could successfully perform with minimal assistance.

The episodes were the units of analysis that were coded in most detail, reflecting their proximity to live teaching actions. The data relating to each episode were individually coded for aspects of MCK that the preservice teacher enacted and thoughts relating to the preservice teacher's MCK related decisions (see Appendix I for an example of the coding). The researcher's coding was checked intermittently by a more experienced researcher and the feedback provided was used to refine the analysis codes and coding process. The following section provides a detailed description of the analysis framework used to discern and categorise aspects of the participants' MCK and their related decisions from the episode data.

3.7 Data analysis framework for coding episodes

3.7.1 Overview of the analysis framework

An analysis framework was designed to code the MCK related actions of the preservice teachers and their thoughts behind those actions for the episode data. Having reduced the data to units based on the preservice teachers' smallest pedagogical goals, the data concerning each episode were analysed in two ways. First, to address research question one, the data were examined for elements that influenced the preservice teachers' decisions about enacting MCK. Second, the type and quality of MCK that was enacted during instruction was coded, to address research question two.

The framework categories were derived from the literature in the first instance and then refined inductively by the researcher as the episode data were coded. The models relating to teacher decision making formed the initial categories used to code evidence of decision making elements (John, 2006; Leinhardt & Greeno, 1986; Schoenfeld, 2010; Shavelson & Stern, 1981; Simon, 1995; Westerman, 1991). The MCK codes within the framework were originally developed from the mathematics education literature (e.g., Ball & Bass, 2009; Ball et al., 2008; Chinnappan et al., 1999, Cuoco et al., 1996; Davis, 2008a; Even, 1993; Harel, 2008c; Hiebert & Le Fevre, 1986; Kilpatrick et al., 2001; Ma, 1999; Mason & Davis, 2013; Mason & Spence, 1999; McCrory et al., 2012; Skemp, 1976; Star, 2005; Wu, 2006). One of the first attempts to code the data was documented (Daniel & Balatti, 2013) and was found wanting. Categories and subcategories in the framework were reconfigured and developed further as coding continued and as codes began to converge and diverge (Patton, 2002). The refined categories and examples of subcategories that were ultimately used to code the preservice teachers' MCK related decisions and actions for all 137 episodes are illustrated in Figure 3 and elaborated in the remainder of this chapter section.

3.7.2 Coding the preservice teachers' decisions for influencing elements

Four aspects of the preservice teachers' MCK related decisions were analysed, three of which are explicitly shown in Figure 3. The first two aspects, classroom circumstances and instructional goals, relate to elements that were discerned for every episode. The third aspect, elements articulated by the participants in their explanations of episodes, refers to any influencing elements (e.g., knowledge, beliefs, instructional constraints, etc.) that could be discerned from the explanations offered by the participants in their episode reflections. The fourth aspect of the preservice teachers' decisions, conflicts, was added after coding began, to capture pairs of conflicting elements that the preservice teachers identified in their reflections. Conflicts are not explicitly shown in Figure 3 because they were present in multiple places in the model. Preliminary analysis of the data revealed that the participants managed competing influences as part of the MCK related decision making process, so additional codes were created to capture the tensions experienced by the preservice teachers. Each of the four decision making aspects of the coding framework is now described in more detail.

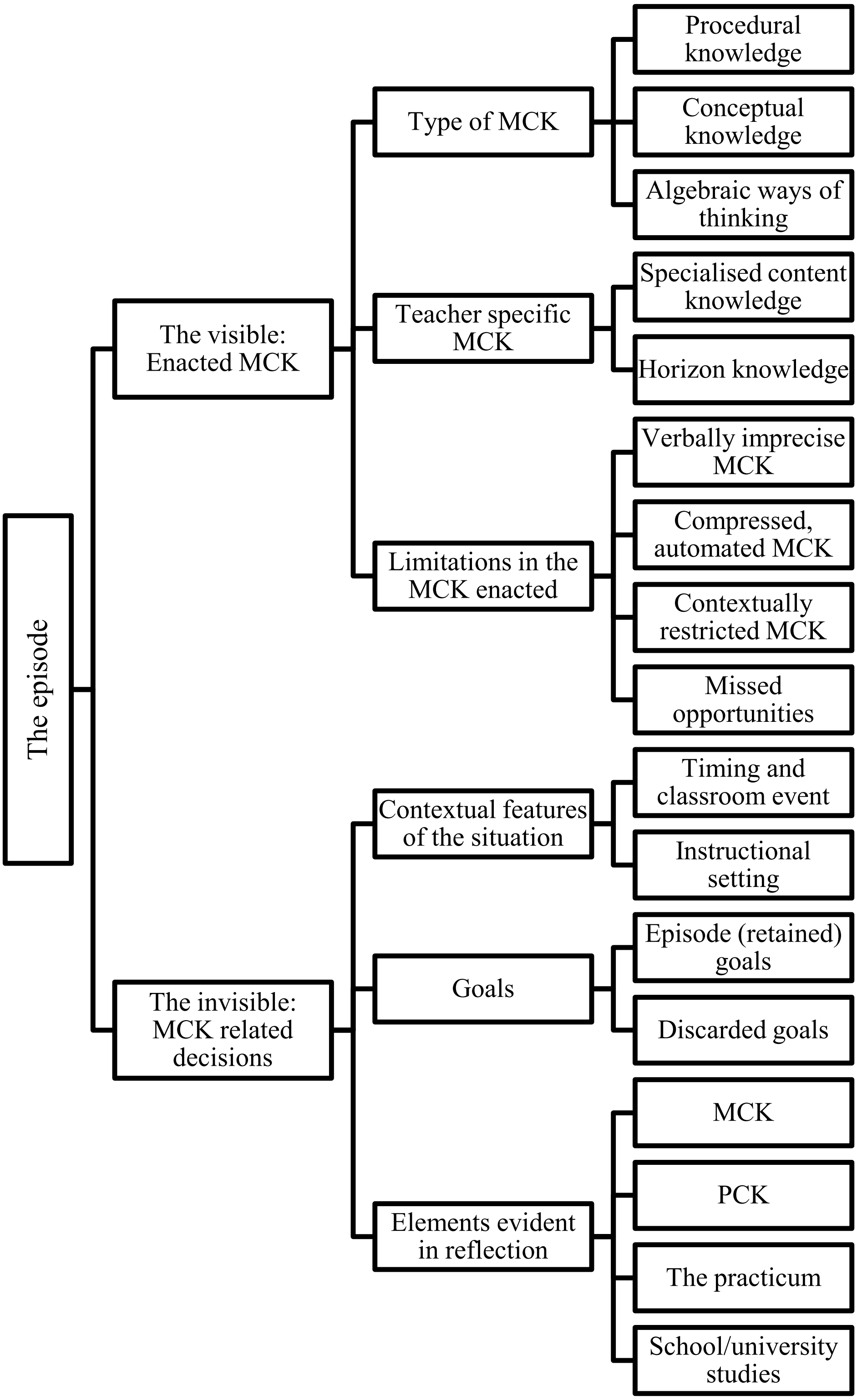


Figure 3. Framework used to code episode data, produced in phase 2 of data analysis

3.7.2.1 Classroom circumstances

Each episode was coded for three elements that concerned live classroom circumstances. Those elements were: (a) the timing of decisions as preactive or interactive, (b) the type of classroom events that triggered the interactive decisions, and (c) the instructional setting. The three-part analysis of how the preservice teachers oriented to live classroom circumstances began with an examination of when the preservice teachers first decided to enact particular MCK. Decisions that were pre-existing, having originated prior to the lesson when the preservice teachers planned their lessons and were then re-established at some point during the lesson were coded as preactive decisions. This reflects Westerman's (1991) finding that decisions made in the "preactive or planning phase" (p. 294) are particularly significant for preservice teachers. Other decisions to enact MCK in an episode can be interactive decisions. These decisions relate to the "interactive or teaching phase" (Westerman, 1991, p. 294). Decisions behind an episode were coded as interactive decisions if they were made spontaneously by preservice teachers during a lesson and did not originate in the preservice teachers' lesson image. Coding each episode as the result of either preactive or interactive decisions provided a distinction between episodes where preservice teachers had had more time to consider their options and those where the situation did not allow the preservice teachers more than a few moments to make their decision. The researcher expected that a reduction in the time to decide on an MCK related course of action may affect the MCK that preservice teachers enacted, and while this did appear to be the case, the effect was not what the researcher had initially anticipated.

Interactive decisions made by the preservice teachers during instruction were heavily influenced by what was happening in the classroom at the time. It was therefore necessary not only to code decisions as interactive ones, but also to code the type of classroom cue (Shavelson & Stern, 1981) that prompted the preservice teachers to make on-the-spot MCK related decisions. Hence, each episode that was coded as the result of interactive decisions was also coded according to the classroom event that the preservice teacher indicated in their interview prompted them to make their decisions.

The categorisation of classroom events prompting the preservice teachers to make interactive decisions was generated inductively. Initial coding of the classroom events produced nine different types of events, such as a student asking a question or a supervising teacher offering advice. The researcher analysed the nine categories for convergence and subsequently nested the nine smaller categories into two larger categories. The first and most frequently coded classroom event category was "student prompted events." In those episodes, interactive decisions were prompted by an event generated by the preservice teachers' students, such as a student question or comment, a student's written work, or even an absence of a response by a student if he or she seemed confused and was unable to answer a question. The second classroom event category, "non-student prompted events" referred to events that took place in the classroom that did not directly relate to student generated events. If, for example, a preservice teacher began to use one form of mathematical notation, and then changed his or her mind, making the interactive decision to replace it with another, the non-student prompted event category was applicable. Non-student prompted events were usually the preservice teachers' own classroom actions which prompted reflection but was sometimes a comment from the supervising teacher.

The instructional setting in which the episodes took place was the third and final indicator used to code how the participants oriented to each teaching situation. Consistent with Simon's (1995) findings that classroom interactions influence live teaching decisions, the number of students (e.g., one individual, two individuals, whole class, etc.) directly involved in the interactions were noted for each episode. After creating the initial coding categories of interaction size, the researcher noticed that the preservice teachers' episode reflections indicated it was not only the number of students directly involved in the interactions that appeared to influence the preservice teachers' decisions. The preservice teachers also referred to the size of the "audience" watching the interactions in their reflections. Taking both the participants and the observers of the interactions into account, four categories of instructional settings were developed, adapted from descriptions of instructional settings in the literature (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Good & Beckerman, 1978; Peterson & Fennema, 1985) and the researcher's field notes. The categories are:

- Whole class (preservice teacher is interacting publicly with multiple students and is aware that all students may be listening in to the discussion);
- Individual in front of whole class (preservice teacher is interacting publicly in a conversation with one student, conducted in front of the whole class, and is aware that all students may be listening in to the discussion);

- Small group (preservice teacher is interacting in a relatively private conversation with between two and five students, while the rest of the class is engaged with independent work and do not appear to be listening to the discussion);
- Individual (preservice teacher is interacting in a relatively private conversation with one other student and believes that the rest of the class is engaged with independent work and do not appear to be listening to the discussion).

The instructional setting of each episode was coded alongside the type of MCK related decisions (preactive or interactive) and, if applicable, the type of classroom event that prompted the preservice teachers to make interactive decisions. The coding of these elements was followed by the coding of a second key element of the decision making process, the preservice teachers' intended goals.

3.7.2.2 The preservice teachers' micro goals

Each episode was associated with one or more pedagogical "micro goals," comprising episode goals and discarded goals. The preservice teachers' pedagogical intent(s), i.e., their goal(s), were present in each episode unit, as the lesson footage and accompanying interview data were organised into episodes using the preservice teachers' goals. A goal in this study is the result that the preservice teacher hopes to achieve and towards which an endeavour (i.e., enacting MCK) is directed. The intended goal or goals behind each episode, referred to in this study as "episode goals", were discerned by the researcher from the interview data, when participants either explicitly described or implied a pedagogical goal that shaped their MCK related teaching actions. The episode goals identified by the researcher were triangulated, where possible, by references in the participants' lesson plans, the lesson footage, and inferences recorded in the observation field notes. In total, the researcher discerned 174 episode goals relating to the 137 episodes coded. The researcher noticed a further 25 goals that were considered by preservice teachers during episodes but were not pursued. Those goals are referred to as "discarded goals" because by choosing not to pursue those goals, the preservice teachers knowingly discarded a particular aspect of MCK that would have been enacted, had they attempted to achieve the goal. Hence, goals formed at the micro level of the lesson comprised 174 episode goals and 25 discarded goals.

The episode goals and the discarded goals were sorted using pattern-seeking techniques (McMillan & Schumacher, 2006). First, the researcher noted similarities among a number of preservice teachers' goals and sorted the individual goals into seventy four categories of similar goal types. The seventy four categories were then inductively clustered into nine broader micro goal categories by "subsuming particulars into the general" (Miles & Huberman, 1994, p. 250). Two of the goal categories are "to develop students' procedural knowledge" and "to avoid student confusion." The micro goal categories are elaborated in chapter 4, the first of the two findings chapters.

At the end of this stage of analysis, all micro goals that were reasonably able to be inferred from episodes were categorised but not critiqued further. The microgoals were taken at face value by the researcher because there was not enough information about the participants' students' mathematical understandings available to critique the goals. Following Schoenfeld's (2011) claim that when making decisions, "people act in the service of the goals they have established by selecting and implementing resources that will enable them to satisfy those goals" (p. 459), the participants' own MCK and other influencing elements that they used to inform their teaching actions were examined.

3.7.2.3 Influencing elements in the preservice teachers' explanations for their decisions

Elements influencing the preservice teachers' MCK related decisions were present in the explanations they provided when reflecting on the episodes. The preservice teachers offered explanations for forming, and occasionally, discarding particular goals for the episodes. They also provided explanations for enacting particular MCK in pursuit of one or more episode goals. The explanations included justifications for their choice of goal and MCK to enact, descriptions of what was on their minds at the time, and accounts of related events that they felt were pertinent to share with the researcher. Justifications, which Buchmann (1987) claims are "a way of assuring that practice will periodically pass muster" (p. 1), were particularly prominent in the preservice teachers' interview responses. It was through the examination of the explanations offered by the participants in their episode reflections that elements were identified as contributing to the preservice teachers' decisions.

An analysis of the elements evident in the preservice teachers' explanations was undertaken in a similar fashion to the micro goals, whereby trends amongst the elements were identified and categorised accordingly. The first parsing of the episode reflections revealed that the preservice teachers not only relied upon their own knowledge, including their own MCK, to inform their decisions but also drew upon external sources of authority, such as their supervising teacher or textbooks. Two broad categories of elements were established: External sources of knowledge (i.e., the textbook or supervising teacher advice) and internal resources (i.e., the preservice teachers' knowledges and judgements). A second parsing of the two categories led to sub-categories being generated. External sources were sorted into two clusters: elements of the practicum context (e.g., the supervising teacher, the class textbook, etc.) and preservice teachers' own school and university experiences. Internal resources were sorted into six categories, using an adaption of the *Mathematical Knowledge for Teaching (MKfT)* framework of Ball and her colleagues (Ball & Bass, 2009; Ball et al., 2005; Ball et al., 2008).

The intent of the *MKfT* framework (Ball et al., 2008) is to describe the mathematical knowledge base of an expert mathematics teacher, so an adaption of the sub-domains of the framework was required to accommodate all of the preservice teachers' explanations. The *MKfT* framework was adapted by retaining the two major domains (MCK and PCK) and six sub-domains (common content knowledge, specialised content knowledge, horizon knowledge, knowledge of content and students, knowledge of content and teaching, and curriculum knowledge) of the framework, but broadening their application to include developing teacher knowledge. For example, the preservice teachers did not provide explanations that directly concerned how mathematics teachers should teach but instead implied what they thought the role of a mathematics teacher was via opinions about what students need. Consequently, the sub-domain, "Knowledge of content and teaching," captured the preservice teachers' judgements about students' mathematical needs and the sub-domain, "Knowledge of content and students," captured explanations that referred to how students understand mathematics.

For the purposes of analysing the preservice teachers' resources, the sub-domains in this study were used to denote teacher knowledge that is both robust and fragile, deep and superficial, reliable and unreliable. The sub-domains within the PCK domain in particular captured questionable preservice teacher "knowledge" that could not be verified and are described in chapter 4. Examples of preservice teachers' explanations that were coded as resources within each sub-domain are provided in Table 9. For two of the PCK sub-domains, examples of relatively reliable and unreliable explanations are included.

3.7.2.4 Conflicts noted in preservice teachers' reflections

The preservice teachers' reflections of the episodes were examined for evidence of any competing sets of influencing elements that they encountered. Different pedagogical directions where MCK was concerned were possible because of the social setting in which the mathematics content and preservice teacher were located. Lave (1988) refers to "the socially organised meaning of 'math'" (p.125), indicating that knowledge in any social setting "has the authority of social validation" (Burton, 1999, p. 23). In a classroom situation, with numerous authors (e.g., supervising teacher, textbook, preservice teacher, school student, etc.) of the mathematics possible (Burton, 1999), tensions can inevitably exist between the different versions of mathematics content that preservice teachers might consider delivering. The preservice teachers had to weigh up whether or not to teach particular aspects of their MCK to their students because not all mathematical knowledge is put to use in a live classroom (Rowland et al., 2009).

Conflicting sets of influencing elements were identified from the explanations that preservice teachers provided in episode reflections. For example, the preservice teachers experienced tension between their own MCK and textbook content when they encountered textbook questions and/or solution paths which did not align with their own mathematical preferences. The preservice teachers had to decide whether to deliver mathematical content authenticated by perceived gate-keepers (Burton, 1999) of the secondary school mathematics community of practice or instead rely upon their own mathematical knowledge. The preservice teachers decided in some circumstances to alter or replace the textbook content. At other times, the preservice teacher placed more weight on external sources of authority, trusting the "cultural consensus" (Burton, 1999, p. 23) and presented content directly from their textbook. When a conflict was noted, the competing elements were coded hierarchically, with the element privileged by the preservice teachers clearly indicated, ensuring the tension and resolving of that tension was captured in the data analysis.

Table 9. Coding internal resources evident in preservice teachers' explanations using the *MKfT* framework

***MKfT* framework (Ball et al., 2008)**		**Preservice teacher explanation in interview**
Domain	**Sub-domain**	
Mathematical content knowledge	Common content knowledge	"And that part there was… when you've taken it from this side here, taken the six, you have to take the six from there." (William, reflecting on subtracting six from both sides of the equation, $a + 6 = 13$)
	Specialised content knowledge	"I realized that this is a different type of question to what they'd been doing in the other ones…x is on both sides here." (Sam, talking about the equation, $16 - x = x$)
	Horizon knowledge	"The vinculum? I just feel, when they get something like this [points to the equation, $\frac{2(x+7)}{3} = 17$], they're used to seeing it. They're not just writing divide by signs." (Sam)
Pedagogical content knowledge	Knowledge of content and teaching (reliable)	"That can be their study resource for when their midterm exam is coming up." (Thomas, reflecting on the glossary of algebraic language he provided to his students; exam is noted on the school term overview)
	Knowledge of content and teaching (questionable)	"If they're creating it [their own equations to solve], I think that's just a good learning experience." (Thomas, reflecting on his request for students to make up an equation to solve; no evidence to verify his claim)
	Knowledge of content and students (reliable)	"When they're working with negatives and positives, they can get quite confused." (Kate, reflecting on student confusion that was evident in the lesson footage)
	Knowledge of content and students (questionable)	"From my experience, more students feel more comfortable doing the substitution method." (Kate, who admits she prefers the substitution method herself; no evidence to verify her claim)
	Knowledge of content and curriculum	"We had specifically just skipped the chapter that was working with algebra and fractions, so I was just thinking in my head, 'We don't want to do fractions.'" (William)

The decision making process, in summary, was analysed with respect to (a) the contextual location of the decision, (b) the goals that underpinned the preservice teachers' decisions, (c) the internal and external resources drawn upon by the preservice teachers, and (d) the presence of competing sets of influencing elements. Complementing the decision making domain of the analysis framework was one used to code the preservice teachers' enacted MCK for each episode.

3.7.3 Coding the preservice teachers' enacted MCK during instruction

The preservice teachers' oral and written delivery of mathematics content during their lessons evidenced their MCK in action. Hence, for each episode, the preservice teachers' verbal utterances and written contributions of mathematics content were coded for the type and quality of the MCK enacted. The researcher also coded the type of MCK that preservice teachers chose to intentionally withhold from an episode, evidenced in 22 episode reflections. Omitted MCK was not judged in terms of its quality, due to its hypothetical nature. To ensure that the process leading to the study findings was as transparent and unambiguous as possible, a multi-layered MCK framework was applied systematically to the enacted and omitted MCK of each episode.

The framework analysing the preservice teachers' enacted MCK was designed to take a measure of the MCK that preservice teachers explicitly shared with their students, with codes included to capture their developing knowledge base. The framework categories were first established following a substantial review of the literature relating to mathematical knowledge for teaching, and more specifically, algebra teaching. The initial framework was used to deductively code data relating to a small number of episodes. While some categories seemed to fit the data well, other categories appeared less suitable. Several iterations of inductive coding, category refinement, and deductive recoding with more episodes followed so that significant observations within the data that were missed by the original framework categories could be successfully incorporated. Eventually, a cohesive and functional MCK framework was finalised and used to code the MCK evident in the 137 episodes. For the teaching actions within each episode, the types of MCK that the preservice teachers delivered were identified and a judgement was made about the quality of the enacted MCK for lower secondary algebra teaching. Each aspect of the MCK framework is described, with brief illustrations from the data.

3.7.3.1 Type of MCK

Three types of MCK captured the preservice teachers' mathematical knowledge: knowledge of algebraic ways of thinking (AWOTS), conceptual knowledge, and procedural knowledge. A fuller description of each knowledge type was provided in chapter 2. The researcher used examples provided in the literature to guide her interpretations of the preservice teachers' actions as evidence of the three MCK types. A summary of typical teaching actions that implicated common content knowledge of each type of MCK is provided in Table 10 and a discussion of specialised content knowledge and horizon knowledge is presented later in this chapter section.

Table 10. Types of enacted MCK with associated teaching actions

Type of MCK	Teaching actions (common content knowledge only)
Procedural knowledge	• One or more steps of a procedure are written either on the board or in a student's book; • A verbal explanation is provided of one or more steps of a procedure, such as solving an equation or evaluating an expression (e.g., "Whatever you do to one side, you do to the other side"); • The presentation of a task to students reflects the preservice teachers' knowledge of a specific procedure required to complete it (e.g., presenting an equation with a pronumeral on both sides to model transposing pronumerals from one side of an equation to another).
Conceptual knowledge	• Verbal articulation of features of mathematical objects, such as algebraic equations, expressions, or terms is provided (e.g., "An equation is made up of two expressions that are equal to each other"); • A verbal reference is made to mathematical symbols and the operations they represent (e.g., explicitly noting the use of a vinculum to represent division).
Algebraic ways of thinking	• Verbal reference to one or more of the following ways of thinking is provided: o Manipulating with purpose (e.g., "I am trying to get x on its own."); o Doing-undoing (e.g., "We use multiplication to get rid of division."); o Algebraic invariance (e.g., "You could write x divided by two [$x \div 2$] as x over two [$\frac{x}{2}$]."); o Building rules from functions (e.g., "Use the addition symbol to represent the word 'sum' in your equation"); o Abstracting from computation (e.g., "How is this expression similar to that expression?").

It could be argued that all episodes evidenced conceptual knowledge and AWOTS of a sort, given that the preservice teachers could all successfully and purposefully manipulate numerical and algebraic objects when performing the procedures. However, the framework refers to conceptual knowledge and AWOTS in teaching actions where that knowledge can explicitly be seen or heard. This qualification aligns with how enacted knowledge has been defined in this study.

3.7.3.2 MCK known uniquely for teaching

The MCK types were coded in the first parsing of the data, without attending to the quality of the MCK evidenced. To gauge mathematical quality, two qualitative perspectives were taken in the second parsing of the data to capture more and less favourable aspects of MCK for a secondary teaching context. Taking a "glass is half full" approach, the MCK enacted in each episode was reviewed for evidence of MCK that went beyond common content knowledge, i.e., specialised content knowledge or horizon knowledge. The researcher drew from the literature wherever possible to differentiate between knowledge types needed for teaching in particular and common content knowledge because the boundaries between the MCK subdomains are indistinct (Adler & Huillet, 2008; Markworth et al., 2009).

The alternative "glass is half empty" perspective was necessary too, to identify limitations of the preservice teachers' MCK that were considered less than ideal for a teacher to present. The qualitative aspects of the framework contributed to a more complete and nuanced picture of preservice teacher MCK within a classroom setting, offering the researcher an opportunity to look for patterns between the presence of certain decision making elements and higher or lower quality enacted MCK.

Enacted MCK was coded as unique for teaching when evidence was available to do so. Ball et al. (2008) claim that aspects of MCK needed specifically for teaching, specialised content knowledge and horizon knowledge, aren't observable in the classroom and for many episodes this was found to be the case. For example, it was not practical to code every episode for evidence of decompressed MCK, a facet of specialised MCK which is described in chapter 2. Every episode showed the preservice teachers having unpacked their knowledge to some degree. Other aspects of MCK needed specifically for teaching, however, provided more fruitful lines of inquiry. The researcher found that it was possible to discern three types of specialised content knowledge in the preservice teachers' actions: knowledge of multiple solution paths, mathematical connections, and mathematical features. Enacted horizon knowledge was rarely evidenced by the preservice teachers during instruction so it was coded as present but not further categorised. The three kinds of specialised content knowledge and the horizon knowledge discerned in the preservice teachers' actions are shown in Table 11, with associated teaching actions.

Table 11. Aspects of teacher specific content knowledge with associated teaching actions

Type of teacher specific content knowledge	**Teaching actions**
Specialised content knowledge: Knows multiple solution paths	• Presents alternative solution paths to one or more students; • Engages in discussions about alternative solution paths suggested by students.
Specialised content knowledge: Knows of mathematical connections	• Explicitly highlights concepts that underpin procedures, procedural steps, or rules; • Explicitly highlights AWOTS that underpin procedures, procedural steps, or rules; • Explicitly highlights connections between mathematical concepts and AWOTS.
Specialised content knowledge: Knows of mathematical features	• Explicitly highlights sub-procedures, nested within larger procedures; • Explicitly identifies and discusses structural features of algebraic objects (e.g., points out that x is now on both sides of an equation, which is different to previous examples).
Horizon knowledge	• Makes explicit connections to more advanced mathematics ideas, related to lesson content; • Makes explicit connections to other mathematics topics, related to lesson content.

Table 11 shows that a range of observed teacher actions provided evidence of specialised content knowledge of algebraic procedures, ways of thinking, and related concepts. Horizon knowledge was enacted by the preservice teachers when they made explicit references to related mathematics content beyond the lesson focus. Decompressed MCK was present in every category because the expectation that MCK could indeed be observed was based on a premise of John Mason's work (Mason & Davis, 2013; Mason & Spence, 1999), that only when one is consciously aware of knowing, can one can share that knowledge with others.

3.7.3.3 Limitations in the MCK enacted

The MCK that preservice teachers enacted in the episodes was coded for five types of MCK limitations. The first four types of limited MCK concern the MCK that the participants did enact and the final MCK limitation concerns additional MCK which should have been enacted in an episode but was instead noticeable in its absence. For the MCK that was presented, four types of limitations are summarised in Table 12, accompanied by relevant teaching actions and pertinent interview reflections.

Table 12. Limitations in MCK with associated teaching actions

Type of Limitation	Teaching actions
Compressed MCK	In the lesson, the preservice teacher does not explicitly connect a particular sub-procedure, concept, or algebraic way of thinking to a related procedure *and* in the interview, the preservice teacher indicated that they had not been able to access the required mathematics content at the time (e.g., "I didn't realise I needed to say …").
Automated MCK	The preservice teacher presents a "one size fits all" approach to procedures that includes unnecessary steps or ignores structural features of an algebraic object (e.g., "Multiply everything outside by everything inside…then collect like terms.").
Imprecise MCK	The preservice teacher uses mathematical language (verbal or written) that is imprecise (e.g., "It's magic!" or, "Get rid of the nine.").
Contextually restricted MCK	The preservice teacher presents a mathematical idea, either verbally or in written form that is limited in its application beyond the immediate mathematical context (e.g., In the context of solving an equation such as $4x - 8 = 40$, the preservice teacher says, "When solving an equation, you must undo addition and subtraction before multiplication and division.").

The first two categories in Table 12 refer to elements of specialised content knowledge that are underdeveloped, while the second two categories in the table concern MCK that, according to Ball and Bass' (2009) descriptions of horizon knowledge, is not suited to teaching.

The researcher sensed that compression of mathematical knowledge (Cohen, 2011) may have been present in many episodes. However, it was only possible to code an episode as featuring compressed knowledge if the preservice teachers offered a reflection in the interview that suggested their knowledge was so compressed at the time, that they were not aware of the mathematical idea they needed to access and enact. If no reflection of this kind was offered by the preservice teacher, the researcher could not be sure if MCK was compressed or simply absent altogether.

The researcher developed the categories during the analysis phase, moving between the literature and the episode data to create and refine each category and subcategory. To categorise verbally imprecise MCK, the researcher drew upon imprecise language reported of practising primary teachers (Heaton, 1992; Hill, Blunk et al., 2008; Sleep & Eskelson, 2012), preservice primary teachers (Zazkis, 2000) and practising secondary mathematics teachers (Smith, 1977) and students (Falle, 2005) to supplement examples of verbally imprecise mathematical language in secondary preservice teacher studies (Dunn, 2004; Rowland et al., 2011).

Five categories of verbally imprecise MCK were used to code the preservice teachers' enacted MCK. Mathematical language was coded as (a) ambiguous, (b) overly casual, (c) non-mathematical, informal code, (d) a maze of tangled words, and (e) used in an incorrect mathematical context. Ambiguous language, noted by Sleep and Eskelson (2012) and Smith (1977), refers to statements such as "all of this" or "multiply it by 20" which fail to clearly designate the mathematical meaning of "this" or "it". Overly casual mathematical language such as "timesing" (Rowland et al., 2011, p. 8), non-mathematical codes such as "get rid of" (Falle, 2005, p. 116) or "plug it in" (Dunn, 2004, 50), and mazes, which comprise a tangle of words without clear meaning (Smith, 1977) all contribute to mathematically weak and imprecise explanations (Dunn, 2004; Falle, 2005; Hill, Blunk et al., 2008; Rowland et al., 2011; Sleep & Eskelson, 2012; Zazkis, 2000). Mathematical language that is used in an inappropriate mathematical context (Heaton, 1992; Hill, Blunk et al., 2008) is the final category of verbal imprecision. Heaton (1992), for example, found a primary teacher's use of the word "reciprocal" problematic because the teacher was using the word to refer to inverse functions rather than the multiplicative inverse of a number. It was also possible to code a statement such as "plug it in" in two ways (ambiguous and non-mathematical code) if more than one type of imprecision was present.

Each episode was coded for notable absences, as well as the four types of limitations in in Table 12. Enacted MCK could be limited by a notable absence of particular mathematical notions that, if enacted, would have improved the mathematical quality of an episode. The researcher identified two very different situations where the MCK that preservice teachers did not deliver was of interest. Firstly, if the preservice teachers intentionally chose not to teach a certain aspect of their MCK, the type of MCK they consciously withheld was coded, using their interview reflections as evidence of MCK they knew and held back. Secondly, if the researcher identified a notable absence of a particular aspect of MCK that the classroom circumstances reasonably called for, specific aspects of MCK that would have enhanced the preservice teachers' instruction were coded as "missed opportunities" for particular episodes.

In summary, the analysis framework for this study was used to identify pertinent decision making elements for the 137 episodes. The framework was also used to gauge the types of MCK enacted by the preservice teachers in the episodes and to judge the suitability of that MCK for teaching lower secondary algebra.

3.8 Data analysis: Phase 3 - Analysing connections

The final phase of data analysis required a synthesis of the decision making and MCK categories, to identify major themes connecting decision making elements and the MCK enacted during instruction. Connections were made in three ways. Firstly, the influencing elements identified during the second data analysis phase were organised into five major categories of influences, reported in chapter 4. Secondly, connections between the presence of certain influences and enacted MCK in the episodes were identified and are reported in chapter 5. Thirdly, chapter 6 reports the connections found between particular influencing elements and stronger and weaker versions of preservice teachers' MCK being enacted. The process used to identify the connections between decision making elements and enacted MCK are now described.

The researcher examined all of the influencing elements identified for the lesson, lesson phase, and episode data for areas of convergence (Patton, 2002). Decision making elements for the episode data were clustered into five influence categories: the practicum context, episode and discarded goals, live classroom circumstances, preservice teachers' MCK, and preservice teachers' judgements about students. The elements coded in the lesson and lesson phase data were then reviewed and added to three of the five categories: the practicum context, episode and discarded goals, and preservice teachers' MCK. To accommodate goals of different grain sizes, the category, "episode and discarded goals," was renamed "goals of the preservice teachers," to include goals at the macro (lesson) level, meso (lesson phase) level, and micro (episode and discarded goal) level of the lesson.

The researcher used pattern clarification strategies (Miles & Huberman, 1994) as the influence categories were developed, ensuring that each influence category included elements from all six participants' episode data. At the end of the first synthesis stage, all decision making elements were accounted for in the five categories of influences, with the exception of two external resources, the preservice teachers' own schooling and university experiences. These resources were not considered by the researcher as significant influences because they were only briefly mentioned by two of the six study participants, compared with the other influence categories, which were repeatedly mentioned by all participants.

The second data synthesis stage involved a series of variable-oriented analyses to identify connections between decision making elements and enacted MCK. A variable-oriented approach (Miles, Huberman, & Saldaña, 2014) to identify recurring patterns in the data that "cut across cases" (p. 103) was chosen over a case-oriented strategy where findings would have been compared for different preservice teachers. This decision was made after a review of the coding revealed many similarities and relatively small differences in the elements influencing the preservice teachers' MCK related decisions and the MCK that they each taught in their lessons. The focus of a variable-oriented analysis is on the interrelationships between variables (Babbie, 2010), which in this study were the elements that influenced the preservice teachers' MCK related decisions and aspects of MCK that the preservice teachers either did or did not deliver.

Using episode data as a common unit of analysis, influencing elements and enacted MCK were already grouped into 137 episodes. Those episodes were partitioned into smaller clusters of episodes, defined by the presence of a particular type or quality of enacted MCK. The presence of one variable (type or quality of MCK) was compared with the presence of other variables (influencing elements or other types of MCK) using NVivo 10 and Microsoft Excel (Microsoft, 2010) software, in accordance with Patton's (2002) premise of "an unruly but surely patterned world" (p. 480). To manage the large number of possible combinations available for analysis, the researcher referred to reflective notes generated during the first data analysis stage. Reflective notes can invite the researcher to take a closer look at variables that appear to go together (Miles & Huberman, 1994) and in this study, alerted the researcher to potentially significant combinations. The analysis revealed the elements and combinations of elements that were associated more often with certain types of MCK, which are reported in chapter 5.

The third and final stage of analysing data for connections produced a model of influencing elements that tend to lead to higher and lower quality MCK of algebra being enacted. Connections between higher and lower quality versions of MCK and associated influencing elements were first identified, then synthesised into a theoretical model. The model, which is presented and described in chapter 6, represents as succinctly as possible, the major conclusions of the study. The establishment of the theoretical model represents the culmination of the analysing connections phase.

3.9 Ethical considerations

In matters concerning people, including children, particular attention to ethical issues is required (Mason, 1996; Punch, 2009). Significant ethical considerations in this study centred around two key issues: (a) the previously established relationship between the researcher and the participants and (b) the collection of live data in school settings. This section describes the issues encountered by the researcher and the steps undertaken to minimise any undesirable outcomes for the participants and their students.

3.9.1 Selection of participants

Additional sensitivity was required of the researcher when undertaking the study due to the nature of the relationship already formed between the researcher and the preservice teachers. The researcher had met and formed a professional relationship with all participants prior to their involvement in the study as the researcher lectured in the mathematics education course which all the participants had completed. Potential ethical dilemmas regarding the preservice teachers' participation in the study and the grades they were awarded by the researcher, their lecturer, for assessment items were identified and processes put in place to ensure that no bias occurred.

The researcher first applied for and received ethical approval to undertake the study at the university where the participants were completing their tertiary studies (approval number H4495). The participants were then invited to participate in the study by the researcher and the researcher's principal supervisor, following one of the mathematics education course classes. This ensured that all potential participants were contacted, fully informed about the project and their involvement and given an opportunity to ask the researcher and/or her supervisor about the project. Creswell (2007) recommends that researchers fully describe the purpose of their study and are careful not to engage in deception about the nature of the study to participants and these recommendations were closely followed by the researcher. The researcher left the room at one point to give the participants an opportunity to ask questions about their involvement in the study to the researcher's supervisor if they wished. Consent forms were completed by the preservice teachers while the researcher was out of the class and were handed to another education lecturer who was independent of the project. The independent lecturer agreed to store the consent forms until the results of the course were finalised, ensuring no bias toward or against any preservice teacher was possible by the researcher for the duration of the course.

The researcher accessed the consent forms after course results were finalised to find out whom of the preservice teachers had consented to participate in the study. All preservice teachers who were invited to participate in the study had agreed to do so and were contacted by email to reconfirm their participation and to organise lesson observation and interview times (Appendix J). In later discussions with the participants, it was discovered that they were generally excited to be a part of the study, expressing a desire for feedback to be provided to them about their teaching by the researcher. Their eagerness to obtain additional feedback on their teaching demonstrated that the perceived benefits to the participants in this case outweighed the risks (Creswell, 2007). The researcher, having spent a semester with the preservice teachers advocating the importance of reflecting on one's own teaching practice, was very pleased with the enthusiasm shown by the preservice teachers but also was mindful of the potential contamination of data if the researcher's comments featured in the stimulated recall interview responses. Feedback was consequently offered to all preservice teachers if they wished to receive any (all participants did) but only after all observations and interviews were completed. Pseudonyms have been used in this thesis, in accordance with the assurance to all participants that their anonymity be protected (Creswell, 2007).

3.9.2 Access to schools, lessons, and students

Schools were chosen as the setting for observations in an effort to gather the most authentic data possible about how preservice teachers enact MCK. The choice of the setting was consistent with a interpretivist research approach but with this choice came several ethical issues. Access to schools and particular classes was needed by the researcher and hundreds of children were to be in the classrooms while observations took place. One state education body and two regional Catholic education bodies were first approached and consent was obtained to approach school staff in the practicum schools. School principals and/or deputy principals, mathematics heads of department, and supervising teachers, also known as gate-keepers (Erlandson et al., 1993; Hill, 2005; Loveridge, 2010), were subsequently approached for permission to undertake the lesson observations (see Appendices K and L) and consent was obtained by the researcher to observe particular mathematics classes that the preservice teachers would be teaching.

Considerations to protect the privacy of the school and students involved and to minimize disruption to the regular class routine were negotiated individually with each school. All requests, such as information and permission notes for parents or checking of lesson footage by school staff, were carried out. The researcher was aware of the importance of gathering data without significantly disrupting the setting (Marshall & Rossman, 2006) and this included not only the data collection periods but also the organisation of the logistics of each class visit. Consequently, times and classes were negotiated with the preservice teachers via email or phone in order to keep the effort required of school staff to a minimum.

The presence of children in the classroom posed ethical issues of consent and confidentiality which were addressed prior to, during, and after each data collection phase. School staff were given an assurance by the researcher via email and phone that the focus of the observation and associated videotaping concerned the preservice teachers' actions and not the students. Further details regarding the assurances given and the placement of the video camera in the room can be found in the information provided to school staff

(Appendix M). It is commonplace in most research that participants are not named (Hill, 2005), nevertheless an assurance was explicitly provided by the researcher that neither the schools nor any students would be identifiable in the research findings. Certain school administrations requested that a letter containing information about the study be provided by the researcher to parents of the students involved. The researcher complied with this request (see Appendix N), providing the relevant preservice teachers with a class set of letters at least one week prior to any observed lessons taking place.

3.10 Ethical considerations of the researcher's role

The role of the researcher in this study was as a participant observer and an interviewer, two research roles that are typical of qualitative inquiry in education (McMillan & Schumacher, 2006). The research position that a researcher takes in any role, however, should be considered as part of research preparations (Punch, 2009). In this study, the researcher's perspective on secondary preservice teachers' MCK related decisions and actions was shaped not only by her reading of the literature but also by her experiences in teaching secondary mathematics and secondary mathematics education. The researcher taught secondary mathematics (Years 8-12) in Queensland for ten years and more recently, lectured and tutored in general education and secondary mathematics education for the past five years. Each of the researcher's roles is described in this section and include references to how the researcher attempted to limit any bias she held (Best & Kahn, 2006) because of her own mathematics teaching experiences.

The role of the researcher throughout all lesson observations was as a participant observer (McMillan & Schumacher, 2006). Non-participant observation is impossible to achieve completely, without the use of a hidden camera or one-way mirror, therefore the researcher took the role of a detached recorder (Burns, 1996) during lesson observations. McMillan and Schumacher (2006) refer to this form of observation as "participant observation" which includes "directly observing and recording without interaction" (p. 346). Following the suggestions made by Lankshear and Knobel (2004) to allow classroom events to unfold as naturally as possible, the researcher relied upon her own experiences as a secondary teacher to choose an unobtrusive position in the classroom. The researcher remained at the back of the room, out of direct view of the students and did not seek to interact with the preservice teacher, other teachers in the room, or any of the students. The researcher's presence and related actions (looking around the room, writing notes, and using the video camera) would have contributed to the dynamics of the classroom context on observation days but every effort was taken, wherever possible, not to disrupt the natural flow of the lesson (Lankshear & Knobel, 2004).

Successful interviews rely upon a strong rapport to be established between the interviewer and the interviewee (Best & Kahn, 2006; Punch, 2009). The researcher had already established a positive rapport with the preservice teachers during the mathematics education course, and it was this rapport, in part, which may have contributed to the preservice teachers' agreement to participate in the study and their willingness to share their thoughts about enacting MCK. Despite a strong professional relationship being in place initially, Spradley (1979) warns that apprehension and uncertainty are often present at the beginning of an interview and some participants may be anxious about the kinds of responses that they feel the researcher needs. The researcher spent time with each of the participants, encouraging the participants to "tell it like it is" and give a brutally honest account of their experiences, explaining that her role was not as their lecturer but as a learner, to understand their experience as fully and accurately as possible. The preservice teachers were reassured before, during, and after data collection that there were no wrong answers and their thoughts regarding all aspects of their lesson to do with mathematics were a welcome and valuable data source.

3.11 Limitations of the study

The decisions of the researcher are scrutinised in this chapter section with particular emphasis on the constraints that those decisions imposed on the study. The limitations associated with the chosen methodological design relate to the choice of participants and mathematics topic, the introspective component of the data, the reduction of data, and the inductive and interpretive nature of the analytic process.

This study examined only six participants teaching similar mathematics content with similar pedagogical approaches. This reduces the generalizability of the findings in three ways. First, the participants shared similar school mathematics backgrounds and tertiary education experiences, limiting the "type" of secondary mathematics preservice teacher investigated in this study. Second, the opportunistic choice of algebra as the topic to be investigated in this study limited the type of content examined in this study. Third, as the analytic framework was derived, in part, from the data collected, other decision making variables that did not feature in this small data set may be significant in a different context. The complexities involved in gaining access to live mathematics lessons during practicum phases at different school locations made it unfeasible to collect data from more preservice teachers, so a broader range of preservice teachers and practicum contexts was not possible. The transferability of the study findings to all preservice teachers, all topics of mathematics, all practicum contexts, or even to algebra lessons, where a different teaching approach is employed is therefore limited. Nevertheless, studying a small number of participants can generate more in-depth information (Patton, 2002). The information provided by the participants in this study contributes to a deeper understanding of the complexities of MCK related decision making and action and shows the value of a small, in-depth study of teaching practice.

Different amounts of data were collected from the preservice teachers, potentially biasing the findings reported. Four participants provided data for two lessons while two participants were only observed teaching one lesson. To ensure that no individual preservice teacher unduly influenced the findings that are reported in chapters 4 and 5, the researcher checked that all study findings were the result of multiple participants' MCK related actions or thoughts.

The approach used to collect and analyse data in the study potentially led valuable data to be lost at a number of points in the data collection and analysis process. The critical incidents that caught the attention of the researcher and the subsequent editing of the video footage created a loss of potential episodes that the preservice teacher may have commented upon. More potential episodes were discarded during the stimulated recall part of the interview because only those episodes that attracted the attention of the preservice teachers were kept for analysis. Because the researcher continually judged whether it was desirable to probe with further questions or to draw the attention of the preservice teachers to particular actions, the data generated may have been distorted. The researcher believed that an interviewer that continually prompted and questioned had the potential to become more of an interrogator, which could have resulted in the participants shutting down or making up in-the-moment thoughts to appease the interviewer (Meade & McMeniman, 1992). Often, it was necessary to stay silent, ensuring minimum interference but missing additional insights. Hence, the researcher privileged what she considered would be more reliable data over a larger data set.

The loss of data was necessary to create units of analysis that included data pertaining to both decision making and MCK. The findings therefore are based on a significant slice of the preservice teachers' instructional practice but not the entire practice. The findings cannot be considered as a comprehensive reflection of all that the preservice teachers did but instead as a snapshot of MCK related decisions and actions that might be found within a preservice teachers' algebra lesson.

The introspective aspect of the stimulated recall technique limits the reliability of the data collected (Lyle, 2003). The preservice teachers' thoughts behind their actions were unavoidably retrospective, as they could not be captured moment by moment during the lesson. It is likely, therefore, that the preservice teachers' articulated thoughts may have occasionally been at best, slightly adjusted or exaggerated thoughts and at worst, hazy approximations or imaginations. Although the data may have been flawed in this way, the preservice teachers' retrospective commentary still offered a version of the participants' thoughts that closely approximated their thoughts at the time. Recorded lesson footage, which has been successfully used to elicit practising and preservice mathematics teachers' thoughts (Artzt & Armour-Thomas, 1999; Muir, 2010; Rowland et al., 2011; Thwaites et al., 2011) was used to stimulate the participants' memories of their teaching. A short time frame between the lesson and interview, recommended by Gass and Mackey (2000), Hurlbert and Heavey (2000), and Lyle (2003) was also observed.

The study findings are limited by the researcher's approach to and analysis of the preservice teachers' MCK. The analytic process employed was necessarily interpretive, so the findings are limited only to the evidence, patterns, and inferences that the researcher was able to discern from the data. Linguistic choices, for example, can provide clues about what one knows about mathematics, according to Meaney (2005), however, she warns that another person's mathematical understanding can only be approximated by interpreting how it is used in particular situations.

Preservice teacher MCK, in this study, is compared with the "ideal MCK" of an experienced mathematics teacher because an MCK framework more suited to pre-novice teachers was not found in the literature. Limitations of preservice teacher MCK, reported in chapter 5, do not necessarily indicate low quality preservice teacher MCK, relative to their stage of development, but only in comparison with expert teachers. Hence, the findings reported in this study offer an indication of relatively stronger and weaker versions of MCK of algebra that preservice teachers hold and how well that MCK compares with expert teacher MCK.

In conclusion, as recommended by Lichtman (2010) and Mason (1996), every effort was made to ensure that the analysis and interpretation process is as transparent as possible. The specific examples provided in the analysis framework, the detailed description of the analysis and synthesis process, and the use of quantitative summaries in the findings in chapters 4 and 5 aim to provide a clear and transparent description of how the findings and conclusions of the study were reached.

Chapter 4: Research findings 1: Influences on preservice teachers' MCK related decisions

4.0 Introduction

This study views the mathematical content knowledge (MCK) that preservice teachers enact (or do not enact) in classrooms as the result of goal-oriented decisions that they make. Those decisions can be preactive, i.e., formed in the lesson planning stages and not made in response to a particular classroom event, or interactive, prompted by an event during instruction. For the purposes of this study, MCK related decisions involve choosing a goal to pursue and choosing particular MCK as a means to realise that goal. MCK related decisions are influenced by a number of elements including the situation in which the preservice teachers find themselves, pedagogical goals that they form, and the knowledge and beliefs that they hold (Schoenfeld, 2011). To better understand why preservice teachers enact particular MCK of algebra in a live classroom setting requires consideration of the elements that led them to make their MCK related decisions. That is why the influencing elements are reported first in this thesis.

The findings reported in this chapter, the first of two findings chapters, contribute to addressing research question 1, reproduced below. The findings describe the influencing elements that impacted the preservice teachers' decisions to deliver particular MCK. Chapter 5 explores the type and quality of MCK that the preservice teachers enacted in their lessons and how the elements reported in this chapter influenced the MCK that the preservice teachers subsequently decided to enact. The conclusions drawn from the two findings chapters together address the research questions of the study:

1. What elements influence the decisions secondary preservice teachers make regarding the mathematical content knowledge (MCK) they enact when teaching lower secondary algebra?

2. What is the mathematical content knowledge (MCK) that secondary preservice teachers enact when teaching lower secondary algebra?

The excerpts of video footage that attracted preservice teacher comment from the ten lessons led to identifying 137 episodes nested in 46 lesson phases. Episodes were defined as a set of one or more teaching actions performed to pursue the goals that the preservice teachers indicated in their interviews. Unless stated otherwise, the "episode" is the unit of analysis. In total, 174 goals were either articulated or implied when preservice teachers spoke about the decisions leading to teaching actions. Of the 137 episodes, 37 were defined by two goals. The findings also include the analysis of an additional 25 goals that preservice teachers considered and then abandoned. The researcher analysed the video footage, interview data, and lesson artefacts pertaining to the individual episodes for evidence of influencing elements impacting (a) the preservice teachers' choice of goal(s) and (b) the preservice teachers' choice of MCK to enact in pursuit of their goal(s). The preservice teachers' reflections of how they prepared their lesson were also analysed for supplementary evidence of influencing elements.

The coding revealed that almost all elements influencing the MCK related decisions concerning MCK could be grouped into five categories. The only exceptions were two elements, discussed briefly in section 4.6.1, that were referred to by one and two preservice teachers, respectively. All six participants contributed to each of the five categories of influencing elements reported in this chapter. For ease of writing, those categories are referred to as influences. The first influence is elements of the practicum context that the preservice teachers indicated they considered when making their instructional decisions regarding MCK. The second influence is the goals impacting MCK related instructional decisions at the macro (lesson), meso (lesson phase), and micro (episode) levels of the lesson. The third influence is the classroom circumstances that were present as the preservice teachers were teaching. The fourth influence is the preservice teachers' existing MCK, including the MCK they acquired in preparation for their lessons. The fifth and final influence is the preservice teachers' judgements on what lower secondary mathematical learners are capable of understanding and how they learn mathematics.

Each influence corresponds to one of three major aspects of Schoenfeld's (2010) model of decision making, described previously in chapter 2. Firstly, Schoenfeld (2010) contends that individuals will consider elements of a situation within a particular context when making decisions. In this study, influence 1 (the practicum context) and influence 3 (classroom circumstances) reflect the preservice teachers' consideration of their situation. Secondly, Schoenfeld (2010) contends that resources (physical, mental, and emotional) are called upon as a decision is made about which goals should be prioritised and how they might be pursued. The preservice teachers' resources which featured prominently in their MCK related decisions are represented by influence 4 (the preservice teachers' own MCK) and influence 5 (judgements about students). Thirdly, Schoenfeld's (2010) premise that decision making is goal oriented is revealed in this study by influence 2 (the macro, meso, and micro goals of the lesson that preservice teachers construct).

Other influences captured in Schoenfeld's (2010) model of decision making and other practising or pre-novice teacher decision making models did not feature in the preservice teachers' reflections. These include contextual elements such as management of student behaviour (Westerman, 1991) and other resources including beliefs about teaching or mathematics (Simon, 1995), access to physical resources in the classroom (John, 2006), the preservice teachers' emotional resources (Schoenfeld, 2010), and the preservice teachers' pedagogical approaches (Leinhardt & Greeno, 1986). These influences may have been present when the preservice teachers made their MCK related decisions, but there was no reference to them in the data.

This chapter examines the five influences (sections 4.1 to 4.5) discerned from the data in two ways. First, the five influences are described individually, using vignettes and frequency tables. Pertinent vignettes accompany the descriptions of the influences to highlight different elements associated with each influence and frequency tables demonstrate the relative prevalence of those elements on the preservice teachers' decisions regarding MCK. Second, the preservice teachers' simultaneous management of multiple influences is examined (section 4.6). Frequency tables and graphs are used to show trends in the presence of one influence (category) when another influence (category) is present. Trends regarding the presence of one influencing element when another element is present are also reported.

4.1 Influence 1: The practicum context

4.1.1 Hierarchy of influencing elements of the practicum context

Lesson and interview data revealed that the practicum context influenced the preservice teachers' decisions regarding the MCK they presented. Influencing elements of the practicum context were evidenced predominantly in the preservice teachers' general descriptions about how they planned their lessons and were also present in the reflections they provided for 21 of the 137 episodes (15% of all episodes). All preservice teachers referred to four elements of the practicum context, namely their supervising teacher, the school term overview provided to them, the mathematical ability of the class according to the school, and the class textbook. The preservice teachers' descriptions also suggested that a hierarchy existed, whereby certain elements influenced their MCK related decisions more than others.

Figure 4 provides an overview of the four influencing elements of the practicum context, ordered from most influential to least influential, in the opinion of the preservice teachers. The preservice teachers' descriptions of each practicum element were compared against their descriptions of other elements of the practicum context. After the preservice teachers' descriptions of each practicum element were compared, the elements were ordered from those considered by the preservice teachers as the most important to bear in mind to those that might be disregarded if desired. The four practicum elements are ordered according to the relative strength of their influence upon the preservice teachers' decisions regarding the MCK they enacted. The elements ranged from those that the preservice teachers took into account with little or no question to elements that influenced the preservice teachers' choice of MCK to some degree, but were generally regarded as discretionary.

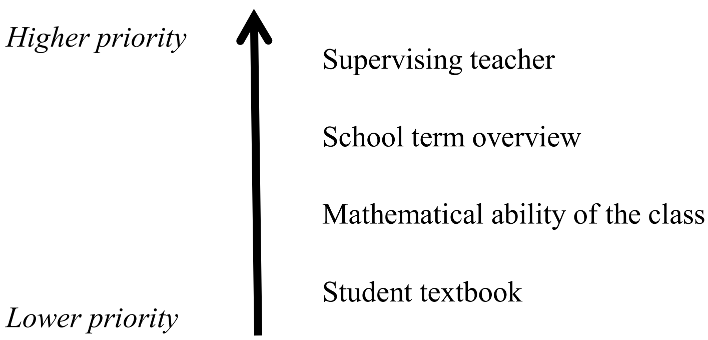


Figure 4. Preservice teachers' prioritising of elements of the practicum context

The elements in Figure 4 reflect how preservice teachers straddle the roles and responsibilities of teacher and student while completing their practicums. Interview reflections about the practicum elements captured the weight of responsibility that preservice teachers felt they carried as teachers, coupled with the experienced student's awareness that their actions would be judged by their supervising teachers. Advice provided by supervising teachers had the strongest influence on the preservice teachers' decisions about MCK, reflecting their status as learners within the practicum context. A term overview was provided to each preservice teacher, outlining weekly mathematical topics to be covered in each class. The preservice teachers were also given a description of their class's mathematical ability (extension, mixed ability, or core) and a copy of the regular textbook used by the class. All preservice teachers referred to their consideration of these elements as they reflected on their MCK related decisions.

The preservice teachers involved in this study indicated that the dual set of expectations placed on them as a result of their roles as teacher and as university student did not impose adverse constraints on their MCK related decision making. The participants did not generally view the elements of the practicum context as inhibiting; on the contrary, they believed that they provided useful guides they could rely upon to help them prepare what they considered were suitable lessons. The influencing elements are now described in more detail and are presented according to the relative strength of each element in shaping the MCK that the preservice teachers ultimately enacted.

4.1.2 The supervising teacher

All preservice teachers indicated that they followed their supervising teachers' advice about presenting particular mathematics content without question or exception. The advice offered by the supervising teachers varied from recommendations relating to mathematical content taught across a lesson to those related to specific mathematical utterances. At the lesson level, Sam's supervising teacher, for example, recommended he reteach how to solve linear equations using the backtracking method (Appendix A, method 1) after he had already introduced his class to the transposing method (Appendix A, method 3) in previous lessons. Sam immediately agreed and introduced the method in the second of his observed lessons. William followed his supervising teacher's more specific advice regarding language to describe operator precedence despite finding it difficult to do so. William reflected in his interview that his supervising teacher had made a comment before his lesson that "she hated BODMAS and all those little sayings." William explained that he had struggled to come up with an alternative phrase for the BODMAS term. During the lesson, he followed his supervising teacher's advice and did not utter "BODMAS" or any related mnemonics but presented what he believed was an inadequate mathematical explanation. Regardless of any difficulties encountered, a sincere attempt to comply with supervising teachers' advice regarding mathematical content was consistent across all preservice teachers.

The possible reasons for the preservice teachers' compliance with their supervising teachers' advice where mathematical content was concerned are two-fold. It may be that the preservice teachers feared negative feedback would accompany any decisions they made that defied their supervising teachers' requests. An alternative reason is that the preservice teachers trusted greatly in their supervising teachers' opinions regarding the mathematical content they should deliver, even more so than in their own. This possibility appears far more likely given the preservice teachers relaxed demeanour around their supervising teachers and the very high regard in which the preservice teachers held their supervisors during the interviews. For example, Ben described the positive influence of his supervising teacher in his interview, saying, "Carl has sort of set me up quite well, to do the type of teaching that I do, because of the way he teaches." The preservice teachers viewed the supervising teachers' content related suggestions as a help and not a hindrance and showed a willingness to take on board their supervising teachers' advice when making MCK related decisions.

4.1.3 The term overview

The term overview was a school document provided to the preservice teachers by their supervising teachers when they began their practicums. The overviews of the 10 week term briefly outlined mathematical topics which the preservice teachers were expected to cover each week of their practicum. For example, Sam's two lessons related to the statement, "One & Two-Step Equations," in his term overview. All preservice teachers indicated that the lessons they presented had been devised using information provided in the term overview. The lessons observed in the study were found to align with the content of the term overviews, highlighting the influence of the term overview on the lesson content the preservice teachers delivered.

The supervising teachers' understanding of the importance of the term overviews strengthened the influence of the overview on the preservice teachers' MCK related decisions. Grace, for example, reflected in her interview, "Barbara [the supervising teacher] just drummed in to me so much, 'You need to make sure you're getting them through this work. You can't fall behind.'" Sam also noted the combined influence of the term overview and his supervising teacher when he commented, “Yeah, she [the supervising teacher] gave me the overview for the term and I sort of came up with a rough plan and then we revised it.” Evidence from the lessons and interviews showed that the term overview, endorsed by the supervising teachers, was an influence on the MCK that the preservice teachers chose to deliver in their lessons.

The term overviews were treated by the preservice teachers as reliable and valuable indicators of certain MCK that they should enact in their lessons. Ben reflected, “It ties you down but…I like having that structure…I like having some idea of where I am going.” The preservice teachers did not describe the school documents as restrictive in nature, stating that they could make slight adjustments to the order of presenting content or solution methods if they chose. Kate commented, “If I had talked to Travis [the supervising teacher] and said, ‘Maybe I want to switch them around,’ I’m sure he would have been fine with that. I guess I just chose it because it was already in place that way.” Kate’s remarks show that although the preservice teachers were free to make some modifications, for the most part they aligned the mathematical content of their lessons with the content descriptions in the term overviews. The influence of the term overviews appeared to be one that the preservice teachers comfortably accommodated as they made MCK related instructional decisions.

4.1.4 Perceptions of the mathematical ability of the class

The class’s perceived ability level appeared to be taken into account where possible by the preservice teachers but not at the expense of other more influential elements. During their interviews, all preservice teachers spoke about the ability level of their classes as a whole, based on school information. The classes were described as either extension classes or mixed ability classes by the participants. However, the preservice teachers privileged other practicum elements, such as the term overview, over considerations of class ability when making decisions about mathematical content, if conflicting influences existed. For example, Grace decided to introduce the substitution method of solving simultaneous equations even though she felt her students were not yet confident with either the elimination or graphical methods. Grace explained that her desire to follow her supervising teacher’s advice not to fall behind and to align the content of her lesson with the term overview recommendations overrode her concerns regarding her mixed-ability class’s capacity to cope with the lesson content. Only when the preservice teachers met what they perceived were more important obligations of the practicum context did they consider their class's mathematical ability when making MCK related decisions.

The preservice teachers' perceptions of their student cohort's mathematical ability did influence their choices regarding the mathematical content they would present. The preservice teachers' interview reflections indicated that with class ability levels in mind, they formed opinions about which aspects of mathematics content best suited their class and varied the MCK they presented accordingly. For example, Thomas decided to introduce his class, an extension class, to a worded problem that could be represented by a linear equation. When he did so, he chose a problem that involved the construction of a two-step linear equation. When the researcher asked if he had considered first presenting a problem that could be represented by a one-step linear equation, he responded with, "My students are above that." By operating with beliefs about student competence in mind, the preservice teachers chose particular aspects of their MCK to enact during their lessons.

4.1.5 The class textbook

The preservice teachers relied upon the class textbook to inform their MCK related decisions but only if the textbook content aligned with the other aspects of the practicum context. All preservice teachers remarked on a desire to use the student textbook but only chose mathematical examples, questions, and solution paths that (a) were related to the content of the term overview, (b) suited the mathematical ability of the class, and (c) were endorsed by their supervising teachers. What follows is an excerpt from Sam's interview where he discusses his use of the class textbook:

> She [the supervising teacher] doesn't, and neither do I really, like a lot of the ways they [the textbook] use to solve it [linear equations] so she uses the same questions and then solves it as she wants them to solve it... That's basically what I did. Because she uses the textbook, I thought, "It's already there, the questions are already there." So I used the same and then changed how I asked them to solve it.

Sam believed that the textbook content was convenient to access and worth presenting in some form but could be adapted to suit other practicum influences when necessary. His inclination to highlight certain textbook ideas, whilst discarding others was echoed by all the other preservice teachers in their interviews. They spoke of accessing content from other textbooks, internet resources, and their supervising teachers or creating their own examples and solution paths to replace what they regarded as less desirable aspects of mathematical content in the class textbook.

The reliance upon the textbook varied among the preservice teachers. In the lessons observed, all six preservice teachers decided to choose certain worked examples, solution methods, or exercise questions directly from the textbook to present to and discuss with their class. Of all the preservice teachers, Thomas used the textbook the least in his two lessons, preferring to create his own examples and questions, which he explained was "to try and make it a little bit more interesting and engage them." In contrast, Sam's reliance on the textbook to inform his MCK related decisions in his first lesson was the most pronounced of all the preservice teachers. As well as presenting questions from his class textbook, he ordered the content of his first lesson using the same sequence of procedures shown in a textbook exercise. For example, he began his lesson using the textbook by asking the class about the difference between an algebraic expression and an algebraic equation. In the interview, he explicitly noted his reliance on the textbook, commenting, "We started this chapter, just talking about the difference between an expression and an equation. And because I'm following the textbook, that was part of the main idea." Thomas' and Sam's uses of the textbook reflect the different levels of influence that the class textbook had on the preservice teachers' decisions about enacting MCK. Both the preservice teachers' interview reflections and their modified use of the textbook content observed during the lessons indicate that textbook content influences the MCK they decide to deliver to a degree, but is not as strong an influence when compared with other practicum elements.

In summary, the MCK that the preservice teachers chose to enact was influenced by four elements of their practicum experience. The preservice teachers complied in the first instance with their supervising teachers' advice regarding mathematical content, followed by the content indicated in the term overview, their perceptions regarding class ability, and finally, the class textbook offerings. The preservice teachers considered these practicum elements in conjunction with other decision making influences as they planned and implemented their lessons. As they did so, they formed goals at multiple levels of the lesson.

4.2 Influence 2: The goals of the preservice teachers

The formation of goals is a key component of teacher decision making (Leinhardt & Greeno, 1986; Schoenfeld, 2010; Shavelson & Stern, 1981; Simon, 2006; Westerman, 1991). A goal, in the context of teacher decision making, refers to a pedagogical aim or intent held by a teacher that both underpins a particular set of teaching actions and describes the desired result of those teaching actions. The preservice teachers formed goals at the macro, meso, and micro levels of their lessons which, in turn, influenced their choice of mathematical content to deliver in their lessons.

Here, the preservice teachers' lesson objectives are defined as the macro goals of the lessons. Working within the practicum context, the preservice teachers created mental images (Schoenfeld, 1999; 2010) of the lessons they would teach and prepared written lesson plans with varying levels of detail. Using the preservice teachers' written lesson plans and their own descriptions of their lesson images provided in the interviews, lesson objectives such as "Simplify algebraic expressions" (Thomas) were identified for each lesson.

Within the lesson objectives, more specific objectives were also identified. Schoenfeld (1999) suggests that observed lessons can be partitioned iteratively into smaller and smaller lesson components that "have a particular kind of structural or phenomenological integrity" (p. 251) and goals can be discerned for these components and sub-components. In this study, the observed lessons were partitioned into lesson phases, defined by the meso goals of the lesson. Phases were further partitioned into teaching episodes, underpinned by the micro goals of the lesson. Chapter 3 included a description of how the preservice teachers' articulated and implied goals for different parts of the lessons were used to decompose each lesson into a series of lesson phases and of episodes nested within the phases.

The meso goals of the lesson were those relating to the preservice teachers' pedagogical intent (e.g., to review mathematical content) in different phases of the lesson. Within the meso goals lay the set of smaller, micro goals (e.g., to address student confusion) that the preservice teachers formed. The preservice teachers either pursued the micro goals by enacting particular MCK within an episode or they knowingly discarded them. This chapter reports on the goals formed by the preservice teachers at the macro (lesson), meso (lesson phase), and micro (episode) levels of the lessons. In chapter 5, findings relating the preservice teachers' goals at each level to the subsequent MCK that they chose to enact are provided.

4.2.1 Macro goals of the lesson: Lesson objectives

As to be expected, the preservice teachers' macro goals influenced subsequent phase and episode goals, impacting the mathematical content they delivered. At the lesson level, the preservice teachers created lesson objectives, developed from the term overview topics. All preservice teachers indicated that their supervising teachers had allowed them to choose their lesson goals as long as they aligned those goals with the general content statements of the term overview. The researcher gleaned the lesson goals from (a) the lesson images articulated in all preservice teachers' post lesson reflections and (b) the preservice teachers' written lesson plans when available. Lesson plans were provided to the researcher for nine of the ten lessons, with Sam's first lesson the only exception. For this lesson, Sam used a textbook exercise as a de facto lesson plan and stated that his goal for the lesson was for his students to be able to successfully master certain procedures in the exercise. Of the nine lesson plans that were produced, lesson goals were explicitly provided in five plans (Kate (two lessons), Thomas (two lessons), and Ben (one lesson)). Macro goals were inferred from details provided in the remaining four lesson plans of Grace (two lessons), William (one lesson), and Sam (one of his two lessons).

Table 13 provides an overview of the macro goals of the observed lessons. The preservice teachers' phrasing of the lesson goals has been modified slightly by the researcher to provide greater clarity regarding the mathematics taught and to more easily highlight similar goals across lessons. The words "familiar" and "unfamiliar" have also been added in parentheses after each macro goal in the column on the far right. This terminology indicates if the preservice teachers' students had already been exposed to the mathematical content and type of procedure in the current algebra unit (familiar) or if the content involved was previously unstudied and therefore unfamiliar to the students.

Table 13. Macro goals of the preservice teachers' lessons: Lesson goals

Teacher	Lesson	Algebra topic	Lesson goals
Kate	1	Simultaneous equations	Solve simultaneous equations using the substitution method (familiar). Represent a word problem by constructing two simultaneous equations (unfamiliar).
Kate	2	Simultaneous equations	Represent a word problem by constructing two simultaneous equations (familiar). Solve simultaneous equations using the elimination or substitution method (familiar).
Grace	1	Simultaneous equations	Solve simultaneous equations using the substitution method (unfamiliar).
Grace	2	Simultaneous equations	Solve simultaneous equations using the substitution method (familiar).
William	1	Linear equations	Represent a word problem by constructing a linear equation (unfamiliar). Solve linear equations using the balance method (unfamiliar).
Sam	1	Linear equations	Classify equations as true or false (familiar). Solve linear equations using the transposing or balance method (familiar). Represent a word problem by constructing a linear equation (unfamiliar).
Sam	2	Linear equations	Solve linear equations using the backtracking method (unfamiliar).
Ben	1	Algebraic expressions, Linear and simple quadratic equations	Solve linear or simple quadratic equations using the balance method (familiar). Represent a word problem by constructing a linear equation (familiar). Simplify algebraic expressions (familiar).
Thomas	1	Algebraic expressions, Linear equations	Identify terms, coefficients, like terms and constants in expressions (familiar). Simplify algebraic expressions (familiar).
Thomas	2	Linear equations	Represent a word problem by constructing a linear equation (unfamiliar). Construct your own linear equation and solve (unfamiliar).

Mastery of procedures dominated the preservice teachers' goals at the macro level of the lesson. The lesson goals presented in Table 13 reflect the emphasis that the preservice teachers placed on mastering algebraic procedures. Chapter 5 explores how their prioritising of lesson goals relating to algebraic procedures appears to contribute to the type of MCK that regularly manifests in their lessons. No preservice teachers offered lesson goals that explicitly referred to the students developing a conceptual understanding of the lesson content or any algebraic ways of thinking (AWOTS) and there was no evidence available from the data to indicate why this was so.

The preservice teachers may have perceived mastery of skills as either a more tangible and therefore measurable goal, or alternatively, as a more important lesson goal than building conceptual understanding or developing ways of thinking. Certain preservice teachers did teach more conceptual knowledge and AWOTS than others in the lesson episodes. The absence of lesson goals relating to conceptual understanding meant that it was not possible to ascertain from the macro goals of the lesson which preservice teachers intended to enact conceptual knowledge and/or AWOTS. However, the goals at the episode level gave a clearer indication of the preservice teachers' intent to include or not include conceptual understanding and ways of thinking in their teaching.

4.2.2 Meso goals of the lesson: Lesson phase goals

Video footage of the observed lessons was partitioned into lesson phases according to pedagogical goals. The pedagogical goals for the phases, which were more fully described in chapter 3, constitute the meso goals of the lesson. The four categories of meso goals captured the preservice teachers' intents to (a) introduce, (b) consolidate, (c) develop, or (d) review algebraic content at different stages of their lessons. This chapter reports on the meso goals that lay behind the preservice teachers' lesson phases and their location within each of the ten algebra lessons.

The video footage that was shown to the preservice teachers in their interviews captured episodes located within 46 lesson phases from the preservice teachers' ten lessons. Each phase was defined by the meso goals the preservice teachers provided either explicitly or implicitly in their post-lesson reflections. On 21 occasions, the meso goals of two adjoining lesson phases belonged to the same goal category (e.g., to introduce new mathematical content). In these situations, the neighbouring lesson phases, nevertheless, were deemed different because their specific meso goals differed. For example, Ben provided eight consecutive meso goals that reflected his intent to review different aspects of mathematical content. His meso goals included reviewing how to solve equations, simplify expressions, and translate worded scenarios into equations. An overview of the preservice teachers' lessons is provided in Table 14, to show how the lessons varied in the type of lesson phases within each lesson and the order in which they took place.

Table 14. Type and relative order of lesson phases in lesson footage

Preservice teacher	**Lesson number**	**Lesson phase and relative order**			
		Introduce content	**Consolidate content**	**Develop content**	**Review content**
Kate	1	5th	3rd	4th	1st , 2nd
Kate	2		2nd		1st
Grace	1			2nd , 3rd	1st
Grace	2		2nd , 3rd		1st
William	1	2nd , 3rd	4th		1st
Sam	1	2nd – 5th	1st , 6th		
Sam	2		2nd – 4th	1st	
Ben	1				1st – 8th
Thomas	1	3rd	4th – 6th		1st , 2nd
Thomas	2	3rd , 4th		5th	1st , 2nd
Total phases (n = 46)		10 (22%)	13 (28%)	5 (11%)	18 (39%)

Table 14 shows that in the video footage of the ten algebra lessons shown to the preservice teachers, there were twice as many phases where the preservice teachers mediated content familiar to their students (consolidate or review content) as phases where unfamiliar content was the focus (introduce or develop content). The majority of the preservice teachers chose to include at least one phase in their lessons where they aimed to review content. The abundance of review phases was due to the inclusion of 'warm up' questions that began all preservice teachers' lessons, with the exception of Sam, who made no comment about the absence of review phases in his two lessons. The preservice teachers also included opportunities for independent practice in most of their lessons, accounting for the relatively large number of phases where the preservice teachers wanted to consolidate their students' understanding of mathematical content.

The preservice teachers made no explicit mention of a link between the meso goals of their lessons and the MCK that they subsequently chose to deliver when they made reference to the phases of their lessons. As chapter 5 will reveal, however, a pattern was discerned between the type and quality of MCK enacted and the phase where the relevant episode was located.

4.2.3 Micro goals of the lesson: Episode goals and discarded goals

The preservice teachers formed two types of micro goals that led them to either enact particular MCK or intentionally omit particular MCK from an episode. The episode is the smallest unit of data analysed in this study. As explained earlier in this chapter, an episode comprises one or more teaching actions involving the preservice teachers enacting an aspect of their MCK (e.g., verbal explanation or written notation) with a specific intention in mind (expressed or implied by the preservice teachers in their interviews). The "specific intention" is defined as the episode goal that prompted a specific set of teaching actions (i.e., the episode). Examples of episode goals that the preservice teachers cited include "to address student confusion" or "to connect procedure with a concept."

In the analysis of the episode reflections, a second type of micro goal, which differed from an episode goal, was discerned by the researcher. These micro goals did not lead to MCK being enacted and are therefore not episode goals. They are goals that were considered by preservice teachers as they weighed up their MCK related options and might have led to particular MCK being enacted but instead, were consciously discarded. For example, four of the preservice teachers considered acknowledging an unexpected procedural suggestion made by students to pursue the goal "value students' mathematical contributions," but instead chose not to do so. Micro goals of this type are referred to in this study as discarded goals because both the goal and the MCK that would have been enacted if that goal had been pursued were intentionally discarded by the preservice teachers. Goals formed at the micro level of the lesson therefore comprise both episode and discarded goals. In total, the preservice teachers commented on one or more episode goals for each of the 137 episodes of teaching in their reflections of the video footage. The preservice teachers' reflections on 22 of the 137 episodes also included references to 25 discarded goals.

The episodes were nested in the 46 lesson phases and the number of episodes in each lesson phase type varied. Table 15 provides an overview of the episodes that did attract attention firstly from the researcher (editing of video footage) and secondly from the preservice teachers (reflecting on video footage), according to the lesson phase in which they were located. The overview indicates the proportion of episodes nested in each of the four lesson phase goal types discussed earlier.

Table 15. Proportion of episodes by lesson phase (meso) goal type

Lesson phase (meso) goal	**Episodes (n=137)**
Introduce content	17 (12%)
Consolidate content	43 (31%)
Develop content	20 (15%)
Review content	57 (42%)

The table shows that in this study, more episodes (57 out of 137 episodes) were located within the review phase of lessons than in any other phase. This over-representation was primarily due to one preservice teacher's lesson (Ben's) being entirely a review lesson and contributing 27 episodes to the total of 57. The lesson phase type with the least number of episodes (17 out of 137 episodes) corresponded to those phases aimed at introducing new content. As no observed lessons were located at the beginning of an algebra unit, the majority of the lessons involved a preservice teacher attempting to consolidate, review, or develop the mathematical content presented in previous lessons of the unit.

Behind each episode lay either one or two episode goals and up to three discarded goals. Table 16 provides an overview of the number of episode goals and discarded goals that preservice teachers referred to in their reflections of the 137 episodes. Reflections of episodes that featured either one, two, or three discarded goals that led to particular MCK being withheld by the preservice teachers are also identified in the table.

Table 16. Distribution of episodes by the number of discarded goals and episode goals

Presence of discarded goals in episode reflection	Episodes		Total
	Single goal episodes	Dual goal episodes	
No discarded goals	85 (62%)	30 (22%)	115 (84%)
One discarded goal	13 (9%)	7 (5%)	20 (15%)
Two discarded goals	1 (1%)	0 (0%)	1 (1%)
Three discarded goals	1 (1%)	0 (0%)	1 (1%)
Total episodes (n = 137)	100 (73%)	37 (27%)	137 (100%)

Table 16 shows the varying number of goals that the preservice teachers managed in different episodes. In their reflections of 85 episodes (62% of all episodes), the preservice teachers commented on one episode goal only. In contrast, one participant commented on four different episode and discarded goals that were managed within a single episode. The collective impact of multiple micro goals on preservice teachers' MCK related decisions is discussed in section 4.6 of this chapter to illustrate the complexity of MCK related decision making. In this chapter section, the two types of micro goals, episode goals and discarded goals, are discussed.

4.2.3.1 Episode goals

Reflecting on the 137 episodes, the preservice teachers either explicitly articulated or strongly implied 174 episode goals that influenced their decisions to present particular MCK. This included 37 instances where the preservice teachers expressed dual goals that collectively influenced their decisions to deliver particular mathematical content. The episode goals were inductively categorised into nine types of goals, summarised in Table 17.

Attention to mastering procedures (category 1) dominated the categories of episode goals shown in Table 17. In some cases, the preservice teachers were quite specific about a procedural step to be covered in an episode. For example, Thomas' intent for providing certain warm up questions was discerned from his interview reflection when he said, "That's why I had these questions lined up. 'Cause I knew they were going to have to expand the brackets."

Table 17. Type and frequency of episode goal categories

Category number	Category of episode goal	Number (n = 174)
1	Develop students' knowledge of procedures	60 (34%)
2	Address student confusion	32 (18%)
3	Teach students appropriate use of mathematical language	16 (9%)
4	Value students' mathematical contributions	12 (7%)
5	Gauge student knowledge	12 (7%)
6	Connect procedure with a concept	12 (7%)
7	Avoid student confusion	11 (6%)
8	Associate procedure with certain types of solutions	10 (6%)
9	Connect procedure with mathematical purpose	9 (5%)

In other instances, a more general reference was made, such as Kate's goal, "To get through the steps," which underpinned her presentation of particular questions to her class. The strong presence of category 1 in the preservice teachers' reflections is examined in chapter 5 alongside the type of MCK that the preservice teachers enacted in corresponding episodes.

Episode goals concerning mathematical connections (categories 6 and 9) were given little attention by the preservice teachers. They rarely spoke of a desire to focus on content that went beyond performing steps of a procedure accurately. The preservice teachers provided only 21 episode goals (12% of all episode goals) about connecting their students' knowledge of procedures with concepts (category 6) or mathematical purpose (category 9) in their reflections. This statistic compares poorly with the preservice teachers' goals regarding the development of their students' procedural knowledge in isolation which were noted almost three times as often (34% of all episode goals). Although all six preservice teachers offered at least one episode goal regarding mathematical connections in one of their lesson reflections, only two preservice teachers, Ben and William, repeatedly aimed to develop connections to either mathematical concepts or purpose.

Ben's interview reflections captured over half of the episode goals where developing connections to mathematical concepts (category 6) were intended. Ben was adamant that his students understand the importance of maintaining equivalence when solving equations. He refused to let his students "just move things across" when they suggested transposing terms from one side of the equation to the other. Ben repeatedly drew a set of balance beams under equations to connect the concept of equivalence with the procedural steps he was modelling, as shown in Figure 5, reflecting in his interview, “The balance beams reinforce the laws of equivalence.”

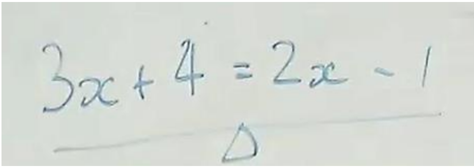


Figure 5. Ben draws a set of balance beams beneath an equation

William was the preservice teacher who repeatedly attempted to make an explicit connection between a procedural step and the mathematical purpose of performing that step (episode goal category 9). In his Year 8 introductory lesson on solving linear equations, William attempted in three episodes to connect “doing the opposite” operation with the effect of reversing the operations already performed on a pronumeral. In his reflection, he commented, “I wanted them to realise that to get the answer they had to do the opposite. So if I was adding the number…to go backwards, to work backwards from the answer to the unknown, they had to do the opposite.” William’s intention to connect a procedural step with its mathematical purpose and Ben’s intention to connect a procedural step with an underlying mathematical concept are examples of the few episode goals that specifically referred to mathematical connections.

The preservice teachers’ opinions about where their students were experiencing or were likely to experience confusion with the subject matter figured in their episode goals (categories 2 and 7). Their opinions resulted in a “fight or flight” response to MCK related goal formation and action. In some cases, preservice teachers decided to “fight” and enact MCK with the specific intent of dealing head on with student difficulties to address student confusion (category 2). For example, Ben spoke of intentionally including operations with integers in his Year 8 review of solving equations. He laughed back at the footage of his students struggling to calculate 12 – 28, commenting, “Oh, I knew they’d struggle with 12 minus 28. They don’t like integers.”

Conversely, the preservice teachers’ concerns about student confusion also led them to choose the “flight” pedagogical option and enact MCK with the specific intent of avoiding student confusion (episode goal category 7). For example, Grace was concerned that manipulating equations with subtraction operations may have confused her students when solving simultaneous equations. She subsequently presented a worked example using equations with only addition operations ($y = 2x + 3$ and $2x + 4y = 5$) in the hope of avoiding student confusion when she began modelling a solution using the substitution method. She reflected, "Well for one, to begin with, they [the equations] all had pluses…where it might confuse people with minuses. I wanted to start with one where they just saw the process." Overall, almost one quarter of all episode goals offered by the preservice teachers related specifically to particular subject matter that they believed could cause their students difficulties.

Preservice teachers occasionally acted to ensure that mathematics was communicated with notation that they considered was the most suitable for a lower secondary, algebraic context (episode goal category 3). The most common symbol that preservice teachers actively discouraged was the use of the obelus (÷) for division, recommending instead that their students use a vinculum. Sam's comment, "I'm trying to get them away from the divided by sign," echoed the sentiments of four of the five other preservice teachers as they aimed to model what they perceived was appropriate mathematical communication.

Two episode goal categories reflected the preservice teachers' intentions to have their students perform mathematics themselves, using their students' preferred methods (episode goal categories 4 and 5). The first of these episode goal categories, "Value students' mathematical contributions," evidenced in 7% of the episode goals, concerned the responsibility preservice teachers felt to allow students to perform procedures using solution paths with which they were most comfortable (category 4). Kate explained, "Yeah, well I tried to go to my way but then when he said that, I said, 'Okay, well…' So I tried to just roll with it." Kate's desire to follow her student's train of thought influenced her decision to teach MCK that aligned with the solution path suggested by a student, rather than teaching MCK of her own preferred solution path.

The second episode goal pertaining to students performing mathematics themselves related to the preservice teachers' intent to "gauge student knowledge" (episode goal category 5). Similar to the presence of category 4, 7% of episode goals were to do with gauging student knowledge. For example, Sam questioned his students at the beginning of his first lesson about the difference between an equation and an expression, commenting in the interview, "I wanted to make sure they still knew." Other preservice teachers looked to identify the presence of particular misconceptions, rather than conceptions. For example, when Thomas asked his students to solve the equation, $5(3x - 12) + \frac{8x}{4} - x + 15 = 3$, his goals included a desire to see if students would struggle to simplify the term, $\frac{8x}{4}$. He reflected, "So the eight x over four…I guess that was where I wanted to help trip them up I guess…see if anyone would pick up whether you could simplify down to two x and then collect like terms."

The preservice teachers very occasionally acted to intentionally shape their students' views about certain mathematical solutions (episode goal category 8), evidenced in 6% of their episode goals. With the "Associate procedure with certain types of solutions" goal in mind, the participants knowingly exposed their students to certain kinds of solutions when performing procedures. For example, Kate chose to present worked examples that she knew would produce negative and fractional solutions, reflecting, "I was just trying to get them to see there's not always nice, even numbers." Kate and the other preservice teachers sometimes made conscious decisions to enact their knowledge of mathematics in a way that influenced their students' expectations about the kinds of solutions that could exist for particular types of mathematical procedures.

The preservice teachers referred to dual episode goals when reflecting on 37 of the 137 episodes (27% of all episodes). Table 18 shows the combinations of goal types that the preservice teachers offered when discussing their MCK related decisions. The table shows a wide variety of goal type combinations underpinning those episodes where the preservice teachers referred to dual goals, such as categories 1 and 2 (five episodes) or categories 4 and 5 (three episodes).

The episode goals that underpin MCK related actions in the episodes are arguably the most direct links between the decision making process and the MCK preservice teachers enact. However, the influence of episode goals on MCK related decisions is not adequately analysed without considering what influences the episode goals themselves. Further synthesis and discussion of the episode goals offered by the preservice teachers is provided in section 4.6, showing how they consider other decision making influences when they form goals that impact the mathematical content they deliver in their lessons.

Table 18. Combinations of episode goal types underpinning the episodes

Episode goal category	Type of episode goal	Episodes with single goal	Episodes with dual goals								
			1	2	3	4	5	6	7	8	9
1	Develop students' knowledge of procedures	42									
2	Address student confusion	19	5								
3	Teach students appropriate use of mathematical language	9	2	4							
4	Value students' mathematical contributions	6	2	-	-						
5	Gauge student knowledge	6	2	-	-	3					
6	Connect procedure with a concept	7	-	1	-	-	-				
7	Avoid student confusion	4	3	1	1	-	-	1			
8	Associate procedure with certain types of solutions	4	3	-	-	1	1	-	1		
9	Connect procedure with mathematical purpose	3	1	2	-	-	-	3	-	-	
Total episodes (n = 137)		100 (73%)	18 (13%)	8 (6%)	1 (1%)	4 (3%)	1 (1%)	4 (3%)	1 (1%)	0 (0%)	

4.2.3.2 Discarded goals

In addition to the episode goals that underpinned MCK related teaching actions, the preservice teachers also spoke of 25 discarded goals in their reflections of 22 episodes. Brief periods of conflict experienced by five of the six participants led them to consider, but then abandon, a goal. Table 16, presented earlier in this section shows that two discarded goals were identified in the reflection of one episode and three in another. Discarded goals were considered by a preservice teacher either at the start, in the middle, or at the end of an episode. Rather than enacting particular MCK to pursue the goal, the preservice teacher instead prioritised other influences and omitted both the goal and hence, the MCK associated with the goal from their teaching actions, as Grace's reflection of one episode illustrates.

Grace commented on an episode in her first lesson where she was modelling a solution to the set of simultaneous equations, $2x + 3y = 2$ and $3x + 5y = 2$, using an explicit teaching strategy in front of the whole class. The episode was located within a phase where Grace was aiming to develop her students' knowledge of solving simultaneous equations by presenting the substitution method to students who were already familiar with the elimination and graphical methods of solving simultaneous equations. Having first solved for the unknown, y, she ended the teaching segment quite abruptly without finding a solution for x and set textbook work for her students to complete independently.

In her interview, Grace referred to a discarded goal that she considered at the time and the priority she gave to other decision making influences instead. She chose to omit the goal "to develop students' procedural knowledge" which she could have pursued by completing the solution and instead chose to save time by not modelling the remainder of the solution to her students. She reflected, "They're still not getting it…but it just takes up so much time and I remember being aware then that I'd gone over what I was supposed to spend on explaining." Grace's consideration but ultimate omission of a goal highlights the presence of conflicting influences when preservice teachers make MCK related decisions, discussed in more detail later in this chapter.

To code the discarded goals, the researcher used four of the nine episode goal categories described previously in this chapter. Table 19 provides a summary of the 25 discarded goals, according to their corresponding goal type.

Table 19. Type and frequency of discarded goals

Type of discarded goal	Discarded goals (n = 25)
Develop students' knowledge of procedures	14 (56%)
Value students' mathematical contributions	7 (28%)
Teach students appropriate use of mathematical language	3 (12%)
Associate procedure with certain types of solutions	1 (4%)

Table 19 shows that the preservice teachers chose to discard goals that focused on developing their students' knowledge of a procedure the most often. Interestingly, this goal type was also referenced the most often by the preservice teachers when they spoke of episode goals. Hence, students' mastery of procedures appears to weigh heavily on the minds of the preservice teachers as they decide to enact or withhold particular MCK.

In total, the researcher discerned 199 goals at the micro level of the lesson from the preservice teachers' reflections of their decisions to enact MCK (174 episode goals) and to withhold MCK (25 discarded goals). The analysis showed that the preservice teachers formed a variety of goals at the micro level of the lesson and managed these goals alongside their macro and meso goals as they made decisions regarding which MCK they should enact (or should not enact) at different points in their lessons.

4.3 Influence 3: Live classroom circumstances

The preservice teachers attended to elements of the live classroom context as they made MCK related decisions in their lessons. Schoenfeld (2010) posits that decision making occurs in response to a person orienting to a particular situation. In this study, the particular situation was the live classroom context. In their reflections, the participants mentioned elements of the live classroom situation that influenced their MCK related decisions. The first element, noted by all six participants was classroom events that prompted them to choose particular MCK to present at certain points in their lessons. The second element, discerned primarily by the researcher and explicitly discussed by two preservice teachers, was the instructional setting in which the preservice teachers were situated. Analysis of lesson and interview data revealed that certain types of classroom events and instructional settings led the preservice teachers to deliver particular types of MCK, a finding that is reported in more detail in chapter 5. A discussion of both the elements and their influence on the preservice teachers' MCK related decisions is presented in this chapter.

4.3.1 Classroom events that captured the preservice teachers' attention

Classroom events contributed to the preservice teachers' decisions to enact certain MCK at different points in the lesson. Shavelson and Stern (1981) state that certain classroom events capture the attention of teachers and teachers make decisions in response to those events. In this study, classroom events refer to actions undertaken by the preservice teachers' students, their supervising teachers, or even themselves that prompted them to make spontaneous decisions about mathematical content during their lessons. Classroom events therefore influence at least some of the MCK related decisions that preservice teachers make during the lesson that they had not made before the lesson began.

The MCK related decisions the participants made were organised into two categories, according to their timing. Preactive decisions (Westerman, 1991) are based on planning undertaken by teachers before the lesson begins while interactive decisions (Westerman, 1991) are in response to a particular classroom event. The creation of a lesson image prior to the lesson allows the teacher to make a number of preliminary teaching decisions regarding the goals they are aiming to achieve and the MCK they are expecting to deliver or withhold to achieve their goals. Those preliminary decisions that are implemented during the lesson and align with the preservice teachers' expectations of how the lesson unfolds are referred to as preactive decisions. Certain classroom events that the preservice teachers had not specifically planned for led the preservice teachers to make interactive decisions regarding MCK. Each episode was coded according to whether it was prompted by preactive or interactive decision making. If interactive, the type of classroom event was also recorded. Table 20 provides an overview of the number of episodes underpinned by each type of decision and classroom event.

When preservice teachers delivered mathematical content during their lessons, they tended to rely upon their preactive decisions, that is, the decisions they had already made before the lesson. Table 20 shows that preactive decisions underpinned two thirds of all episodes.

Table 20. Types of decision making and classroom events underpinning episodes

Type of decision making	Classroom event	Episodes (n = 137)
Preactive	-	92 (67%)
Interactive	Student generated	37 (27%)
Interactive	Non-student generated	8 (6%)

In those reflections, the participants implied that the lesson had progressed as they had expected and the mathematical content presented in the episode was what they had planned to deliver prior to the lesson. For example, Grace emphasised her use of brackets when explaining the substitution method for solving simultaneous equations to her students. In her interview, she gestured to a statement that she had highlighted on her lesson plan, commenting, "I just wrote it in my notes. See here…, 'Remember brackets when subbing in.'" Episodes such as these were coded as resulting from preactive decisions.

The remaining one third of episodes involved the preservice teachers responding to a particular classroom event by establishing impromptu goals and choosing particular MCK to enact to satisfy those new goals. The classroom events that prompted the interactive decisions were also categorised according to whether they were student generated or not.

Student generated classroom events contributed to the majority of interactive decisions related to MCK. The participants did not refer to student generated events concerning behaviour management issues but rather their responses to the mathematical content presented in the lessons. The most common type of student generated event involved a verbal comment from a student (19 episodes), followed by written work (9 episodes), verbal questions (6 episodes), and body language (3 episodes). All preservice teachers indicated in their interviews that particular events involving their students had, at times, influenced their decisions to share particular mathematical content. Thomas, for example, commented on a mathematical discussion he undertook in his second lesson with one of his students who raised his hand for help to solve the equation, $\frac{x}{5} - 9 = -6$. In his interview, Thomas gestured to the term, $\frac{x}{5}$, and stated, "He was struggling with this expression… He was like, 'What do I do here?'" The preservice teachers interpreted student generated events as an indication of their students' level of comprehension with the lesson content and made interactive MCK related decisions with those indications in mind.

Classroom events that did not involve student contributions led to fewer interactive decisions regarding MCK. The interactive decisions that were prompted by non-student generated events involved the preservice teachers reconsidering the mathematical content they had delivered up to that point in the lesson, either on their own or with the assistance of their supervising teachers, and consequently deciding on new goals and/or new MCK to enact to achieve their goals. For example, William had just written the expression, $c \div 2$, on the board when he rubbed it out and replaced it with the expression, $\frac{c}{2}$. In his interview, he reflected that having written the initial form of notation on the board, which was an intentional and planned replica of the textbook notation, he had paused for a moment and looked at what he had just written. He then changed his mind, rewriting the expression without an obelus and telling his students, “That’s really naughty. Don’t do that.” Only four of the six preservice teachers made interactive decisions without being prompted by student behaviours and they rarely did so, indicating that classroom events of this kind did not regularly influence the preservice teachers’ MCK related interactive decisions.

Classroom events that capture the attention of preservice teachers can not only prompt them to make interactive decisions to teach MCK but also, to intentionally withhold MCK from their students. When five of the six participants made decisions to intentionally omit an aspect of mathematical content from their students, the decisions tended to be interactive ones (17 out of 25 decisions to withhold MCK), prompted by a classroom event, rather than preactive decisions, which dominated the decisions to enact particular MCK. The types of classroom events, namely student and non-student generated, that led to the preservice teachers deciding to omit mathematical content were the same as those that prompted the preservice teachers to enact certain MCK. However, the classroom events which influenced interactive decisions to omit MCK were more evenly distributed between student generated (8 decisions) and non-student generated (9 decisions) events than those events that led to the preservice teachers deciding in the moment to enact MCK. These results suggest that classroom events that capture the attention of preservice teachers can lead not only to interactive decisions to deliver MCK but also, to intentionally withhold MCK from students.

The classroom events that compelled the preservice teachers to make interactive decisions concerning MCK are an important influence to examine because they seem to elicit qualitatively different MCK from the MCK that results from preactive decisions. Chapter 5 reports on patterns discerned between the type of classroom event that prompted interactive decisions and the MCK presented by the preservice teachers as they responded to the events.

4.3.2 Instructional settings

In this study, the instructional setting refers to the number of students that the preservice teacher perceives are participating in, or listening to, the interactions taking place in an episode. The preservice teachers rarely mentioned the instructional setting as an influence when they reflected on their lessons. However, the researcher discerned a pattern between the type of MCK that the preservice teachers chose to enact and the instructional setting in which the MCK was enacted, which is described in chapter 5. Hence, the instructional setting appears to be an influence. The researcher's observations indicated that in the episodes observed, preservice teachers delivered MCK within four instructional settings. Table 21 shows the prevalence of each in the set of 137 episodes.

Table 21. Instructional settings used in episodes

Instructional setting	**Episodes (n = 137)**
Whole class	116 (85%)
Private discussion with one student	10 (7%)
Discussion with one student conducted in front of the class	8 (6%)
Small group (2-5 students)	3 (2%)

Instructional settings involving whole class cohorts dominated the episodes examined in this study. Eighty-five percent of all episodes involved a preservice teacher engaging with multiple students in front of the whole class. Of the remaining 15% of episodes, 6% involved public interactions between a preservice teacher and an individual student conducted in front of the class, where the preservice teacher was aware that the remainder of students were able to watch and listen to the interaction. The final 9% of episodes involved relatively private conversations between a preservice teacher and either one individual (7% of episodes) or a small group of students (2% of episodes).

Two preservice teachers explicitly referred to the instructional setting when reflecting on decisions to withhold particular MCK in class. Ben, for example, was conducting a whole of class explicit teaching sequence when he appeared to ignore a student who commented in front of the class that he had solved a question using a different method to the one Ben had just presented. In the interview, Ben reflected on intentionally choosing not to engage in a mathematical discussion with the student in front of the whole class. He commented, "I've had this problem before where I'll try and deconstruct what he says…He would have gone some abstract way about it… I didn't want to engage in that with the [whole class]." Ben's opinion of how his class would have made sense of the hypothetical discussion taking place in front of them influenced his decision to avoid teaching particular MCK during one of his episodes. Grace also provided similar reasons for withholding MCK during one of her episodes. Their reflections suggest that decisions regarding MCK are shaped, in part, by the preservice teachers' perceptions of the students listening in at the time and the mathematical content those students should be exposed to or can successfully comprehend. Chapter 5 explores how the preservice teachers' consideration of these instructional settings appears to impact the MCK that they decide to enact.

4.4 Influence 4: Preservice teachers' existing MCK

The preservice teachers' own MCK was a significant influencing presence in their thoughts as they made their decisions to enact MCK. An analysis of the preservice teachers' interview reflections indicated that their own MCK influenced their decisions in two ways. Firstly, the preservice teachers drew on their MCK to form the goals that underpinned their episodes. Secondly, the preservice teachers drew on their MCK to make decisions about what mathematical content they should deliver to achieve their goals. In this section of the chapter, the MCK discerned from the preservice teachers' reflections of episodes is described. Following this discussion, findings about the preservice teachers' perceived adequacy of their own MCK are presented. These findings include a description of the preparations that the preservice teachers undertook prior to the lesson regarding their MCK.

4.4.1 Preservice teachers' MCK of algebra

An analysis of the interview transcripts revealed that the preservice teachers regularly drew upon aspects of their own MCK to rationalise their MCK related decisions. In their reflections of 74 episodes (54% of all episodes), MCK was discerned in the reasons preservice teachers offered for forming or discarding particular goals and either enacting or withholding MCK to achieve those goals. The MCK discerned from the explanations was categorised using the components of the researcher's MCK analysis framework, which comprised common content knowledge, specialised content knowledge, and horizon knowledge. The components were adapted from the *Mathematical Knowledge for Teaching (MKfT)* framework (Ball et al., 2008), described in detail in chapters 2 and 3. Those aspects of MCK that featured a degree of specialisation for teaching, namely specialised content knowledge and horizon knowledge, were distinguished from aspects that comprised common content knowledge where no specialisation for teaching was noted. The MCK evidenced in the preservice teachers' reflections for the 74 episodes is summarised below in Table 22.

Table 22. Type of MCK evident in episode reflections

Type(s) of MCK	**Episodes (n = 74)**
Specialised content knowledge	40 (54%)
Horizon content knowledge	5 (7%)
Specialised content knowledge and horizon knowledge	1 (1%)
Common content knowledge	28 (38%)

The preservice teachers drew on specialised content knowledge the most often, followed by common content knowledge, and only occasionally, horizon knowledge. The preservice teachers called on specialised mathematical knowledge for teaching when they sought to explain their use of alternative notation or solution paths, particular mathematical questions, or procedures. Ben, for example, asked his class to solve the equation, $4x^2 = 16$. In his interview, Ben justified his choice of equation by explicitly referring to his intentional inclusion of a squared pronumeral. He explained that "the square issue" was a feature of the equation that increased the mathematical complexity of the solution, when compared with the linear equations that he usually presented. Ben's specialised knowledge of particular features of equations and the relative increase or decrease in mathematical complexity of the ensuing solution path was discerned from his episode reflection. Similar explanations were provided by the other preservice teachers, suggesting that they possess a developing knowledge of mathematics specifically for teaching, which influences their MCK related decisions.

Common content knowledge was also a significant influence on the MCK related decisions that the preservice teachers made. Kate, for example, presented a worded question about mathematics and English test results to her class. On the board, Kate used the pronumerals m and e to represent the two test results, rather than the pronumerals, x and y, which were provided in the same worked example of the class textbook. As she wrote up the pronumerals on the board, Kate said to her students, "We don't have to do that. Why don't we make it easier? Why don't we say m is going to equal our results for maths." In the interview, Kate reflected on what she meant by "making it easier" when she spoke to her students, repeatedly justifying her actions with her own mathematical preferences. Her reflection, with underlined passages highlighting her preferred mathematical approach, is as follows:

> When I say make things easier, <u>it's because in my mind, they are</u>, especially <u>when I'm going through</u> variables or equations. <u>Like I do</u>, even now, if it's being represented by depth or length, <u>I'll put a d, I'll put an l</u>, just so <u>it's a way of reminding me</u> sometimes when there are so many x and y's, <u>I get confused</u>. What does the x stand for? What does the y stand for? <u>My reasoning</u> was m would be math and e would be for English.

Kate drew only on common content knowledge in her episode reflection. She presented knowledge that letters related to a worded problem (i.e., m for a mathematics test result) can be used to represent pronumerals. However, there was no evidence of any specialised knowledge of pronumerals, equation structure, or procedural features for a teaching context in her justification for using different pronumerals to the textbook. Kate's common content knowledge, described by Ball et al.(2005) as "basic skills that a mathematically literate adult would possess" (p. 45), in this case was a stronger influence than the textbook content, on her decision to present particular mathematics content in the episode. The preservice teachers' MCK that influences their related decisions can be a reflection of the preservice teachers' own mathematical preferences that are not necessarily specialised for the work of teaching.

4.4.2 Preservice teachers' perceptions of their MCK

All preservice teachers indicated in their interviews that they held sufficient mathematical knowledge to cope with the demands of their lessons. The preservice teachers generally exuded confidence when they spoke about the adequacy of their own mathematics knowledge for teaching junior secondary algebra and only occasionally voiced concerns about knowing too much mathematics and not too little. All preservice teachers implied that they were thinking about algebra from such an advanced perspective that they sometimes found it difficult to explain simple procedures with clarity. William reflected at one point, "When it's so basic is when I struggle." Possessing so much MCK appeared, in the opinion of the preservice teachers, to be making life more difficult in the classroom so it is perhaps not surprising that no preservice teachers considered taking steps to increase their own knowledge of the content as part of their lesson preparations, unless the practicum context required them to do so.

Influencing elements of the practicum context provided the only impetus for preservice teachers to expand their MCK prior to the lesson. Sam was the only preservice teacher to intentionally develop his MCK when preparing both his lessons so that the content he delivered in his lessons aligned with the textbook content and advice from his supervising teacher. When planning his first lesson, he taught himself a procedure he noticed in the textbook used to classify an equation as true or false before modelling the solution of the following problem during his lesson: "If $x = 2$, state whether the equation, $7x = 8 + 3x$ is true or false." Sam reflected, "It's not something I'd seen before but it's just working out if the left-hand side is equal to the right-hand side." For Sam's second lesson, he learned to perform a modified version of the backtracking method for solving linear equations, at the request of his supervising teacher. From Sam's and others' comments, it appeared that the preservice teachers were gauging their required level of MCK from their capacity to perform the procedures inherent in the term overviews, the textbook chapters and the supervising teachers' requests. Only if a procedure was not already part of their repertoire did they consider developing their MCK in any way. Overall, the preservice teachers seemed convinced that their own MCK was above and beyond what they might need for such "basic" algebra lessons. Therefore, the preservice teachers' existing MCK, with rare modifications, formed the mathematical knowledge base that they drew from to make MCK related decisions in their lessons.

4.5 Influence 5: Judgements preservice teachers hold about students

The preservice teachers' pedagogical judgements about their students were a significant influence on their MCK related decisions. Preservice teacher reflections revealed that their knowledge of and beliefs about their students influenced the decisions made about the mathematical content to deliver or withhold from different students. The preservice teachers' actions and reflections also revealed that their judgments regarding students were, at times, questionable and reflected their standing as inexperienced mathematics teachers.

4.5.1 Preservice teachers' judgements about mathematics students

The preservice teachers referred to their judgements about students when they explained one or more MCK related decisions in their reflections of 92 episodes (67% of all episodes). The preservice teachers' student judgements that influenced their decisions to enact MCK mainly involved thoughts about how students learn mathematics and what students need to learn mathematics successfully. For example, Ben shared the judgements he held about students when he explained his decision to present a reminder about the distributive law (Figure 6) on the class whiteboard as his lesson began.

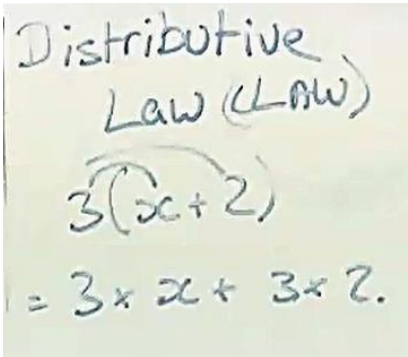


Figure 6. Ben's written reminder about the distributive law

During the stimulated recall part of his interview, Ben pointed to his board work and explained why he had delivered this aspect of his MCK. He first explained that the content was one of three mathematical ideas that had been "repeatedly coming up as issues" for his students in past lessons. He then reflected on the value of "repeatedly saying the same things in lessons," implying that repeated exposure to mathematical content was an appropriate way for students to learn mathematics. Ben's judgements of his students' understanding of the content and his opinions regarding how students learn and the content to which they should be exposed influenced his decision to present particular mathematics content in the episode.

The preservice teachers' reflections exposed a wide range of judgements about how students learn mathematics with differing mathematical emphases. Grace, Thomas, and Sam offered explanations implying that students learn mathematics well when teachers present content in small, manageable steps and ensure that students have time to practise those steps to develop mastery of the procedures. In contrast, Kate, William, and Ben spoke about "getting them to understand [a] concept" (Kate), implying that students learn mathematics well when conceptual understanding is developed. The preservice teachers also provided a number of other judgements to do with how students learn mathematics. The judgements varied from perceived student strengths and weaknesses to student preferences and aversions. The preservice teachers' diverse range of student judgements impacted the MCK enacted by influencing their decisions regarding if and when it was beneficial for their students to be exposed to particular mathematical content.

4.5.2 Researcher's perceptions of the preservice teachers' judgements

The preservice teachers interviewed appeared confident with the reliability of the judgements they held about how students learn mathematics in general or about how specific students learn. In their interviews, the preservice teachers did not refer to any possible limitations that their pedagogical judgements about their students may have had. This is despite their lack of any experience in teaching lower secondary algebra in formal classroom settings prior to their current practicum.

The preservice teachers' reflections that involved judgements about students were cautiously categorised using the pedagogical content knowledge (PCK) components of the *MKfT* framework (Ball et al., 2008). It was not assumed that the "knowledge" that the preservice teachers called upon to inform and explain their MCK decisions was accurate or robust, which is the original intent of the framework. Rather, the pedagogical reasons involving students that the preservice teachers offered in their interviews were a combination of substantiated and unsubstantiated knowledge and beliefs dominated by two of the PCK subcategories in the framework, namely "knowledge of content and teaching" and "knowledge of content and students". Table 23 summarises the preservice teachers' student judgements that were evidenced in reflections they offered for their MCK related decisions, and the component of PCK (Ball et al., 2008) that was most closely associated with each type of judgement. The judgements comprised reasonable or questionable knowledge relating to (a) how students should be taught mathematics, (b) how students learn mathematics, and (c) whether students should be exposed to particular aspects of mathematics content, given the students' year level and the associated mathematics curriculum (identified from the term overviews or textbooks).

Table 23. Type of student judgements evident in episode reflections

Judgements about students (mathematically speaking)	**Associated component of PCK (Ball et al., 2008)**	**Episodes with reflections featuring student judgements**	
		Reasonable judgements	**Questionable judgements**
How students learn	Knowledge of content and students	28	22
Students' needs	Knowledge of content and teaching	15	11
How students learn and students' needs	Knowledge of content and students, knowledge of content and teaching	4	10
Content to which students should be exposed	Knowledge of content and curriculum	2	-
Total episodes (n=92)		49 (53%)	43 (47%)

Table 23 shows that almost half of the explanations offered by the preservice teachers about students were questionable. The researcher did not have enough knowledge of individual students or class cohorts to verify the questionable explanations and justifications offered by the preservice teachers. Thus, for many of the preservice teachers' episode reflections, it was not possible to ascertain where evidence-based knowledge ended and speculation began.

In some circumstances, evidence contradicting the preservice teachers' judgements was present. For example, Kate's justification for using the pronumerals m and e in a worded question to represent the results of a mathematics and English test respectively, rather than the textbook's choice of the pronumerals, x and y, (discussed in section 4.4.1) was that it would be easier for students. However, Kate's students did not react positively to her use of the pronumerals, m and e. After persevering for a few minutes with the idea, Kate eventually agreed to her students' repeated requests to use the pronumerals, x and y. In this situation, Kate's assumption about how her students would respond to particular content appeared to be incorrect. In contrast to this example in which the preservice teacher was aware of a misjudgement, there were other questionable judgements that the participants believed were quite reasonable.

The researcher, an experienced secondary mathematics teacher, identified a number of questionable statements offered in every interview. Explanations offered for decisions such as "They don't like it" (Ben) and "I don't think they would have understood that" (Grace) could not be substantiated by the video footage or interview comments and appeared to be possible examples of preservice teachers misjudging student needs and abilities. Other judgements appeared, to the researcher, to be broad generalisations of mathematics students being applied to the class, such as Sam's comment, "I wouldn't use the word substitute with Year eights." Those explanations led the researcher to question the reliability of their judgements where students were concerned.

Despite the questionable nature of the judgements regarding their students, the influence of the preservice teachers' judgements about students on their MCK related decisions was significant. The impact that these often questionable judgements appeared to have on the type and quality of MCK that the preservice teachers ultimately enacted is explored in chapter 5.

In summary, the data analysis revealed that the five influences that impact preservice teachers' MCK related decisions are: (a) the practicum context, (b) goals formed at the macro, meso, and micro levels of the lesson, (c) live classroom circumstances, (d) the preservice teachers' own MCK, and (e) judgements that preservice teachers hold about students. The researcher noted that not all influences appeared to be equally as important in different circumstances and particular combinations of influences seemed to be clustered together in the preservice teachers' episode reflections. Hence, a qualitative cross-variable analysis was undertaken of the influences involved in the decisions that preservice teachers made for each episode. Trends regarding the presence of certain influences when other influences were also present were identified and are reported in the final section of this chapter.

4.6 Reading across decision making influences

When preservice teachers decide on the MCK they should teach, they weigh up a number of influences simultaneously. In this culminating section, the complexity of MCK related decision making is indicated by an examination of how the influences described in sections 4.1 to 4.5 collectively impact the preservice teachers' decisions. This section is presented in two parts. First, combinations of the participants' self-reported influences for the episodes are described. Second, combinations of influencing elements discerned by the researcher are described. In both sections, the combinations of influences and influencing elements that were associated with particular types of MCK are reported and are revisited in chapter 5.

4.6.1 Combinations of influences evident to the participants

This section describes the preservice teachers' self-reported influences on their MCK related decisions in four ways. To begin the section, the preservice teachers' micro goals are synthesised into two major goal categories and combinations of these two goal categories for the 137 episodes are presented. The combinations of influences discerned from the participants' explanations for their choice of micro goals and their choice of MCK are then explored. The section concludes with a discussion of how the participants managed competing combinations of influences.

4.6.1.1 Reading across micro goals

Micro goals, formed and either pursued or discarded when MCK related decisions are made, are a critical influence on the MCK that preservice teachers present. Preliminary analysis of the micro goals offered by the preservice teachers in their interviews revealed that they formed (a) episode goals which underpinned the MCK they subsequently enacted and (b) discarded goals which resulted in omitting particular MCK. Nine categories of micro goals were reported and described in section 4.2.3. The secondary analysis reported here involved reading across the micro goals to discern any connections between those nine categories. Major categories of micro goals were developed inductively by the researcher by considering broader themes that appeared to be most strongly associated with the goals within each of the smaller categories.

The analysis produced two major categories of micro goals. The first was "Content focused goals," characterised by the preservice teachers' intent to deliver particular mathematical content, without making any reference to students' associated knowledge. Goals in this major category explicitly referred to skills or aspects of mathematical knowledge that preservice teachers felt were important for their students to learn. The second major goal category, "Student focused goals," referred to the preservice teachers' intent to align the content they delivered with their students' understanding of mathematics. Despite enacting MCK in each instance, the preservice teachers did not specifically refer to a desire for their students to acquire particular mathematics content or skills when they offered goals in this major category. Instead, the preservice teachers explained that they were delivering MCK in an attempt to align the mathematics content of the episode more closely with how their students currently understood the content. Table 24 provides an overview of the two major categories of micro goals.

Table 24. Categorisation of micro goals into content and student focused goals

Major category of micro goal	Episode goals (n = 174)	Discarded goals (n = 25)	Total (n = 199)
Content focused goals	107 (61%)	18 (72%)	125 (63%)
Develop students' knowledge of procedures	*60*	*14*	
Teach appropriate use of mathematical Language	*16*	*3*	
Connect procedure with a concept	*12*	*0*	
Associate procedure with certain types of Solutions	*10*	*1*	
Connect procedure with mathematical purpose	*9*	*0*	
Student focused goals	67 (39%)	7 (28%)	74 (37%)
Address student confusion	*32*	*0*	
Value students' mathematical contributions	*12*	*7*	
Gauge student knowledge	*12*	*0*	
Avoid student confusion	*11*	*0*	

The summary in Table 24 shows that when preservice teachers made decisions to enact particular MCK, they formed goals that focused on the content they wanted to teach (61% of episode goals) more often than on the students they were teaching (39% of episode goals). Interestingly, they were also prepared to discard more content focused goals than student focussed goals when conflicting influences presented themselves (conflicting influences are discussed in section 4.6.1.4). The findings may reflect the preservice teachers' broader intents to ensure that all of the lesson content is covered, the lesson goals are achieved, and they keep up with the term overview content sequence.

The goals that led to MCK related teaching actions occurred separately and together when preservice teachers made their decisions. Underpinning the 137 episodes in the study lay either one single episode goal or two compatible episode goals. Table 25 shows the presence of the two overarching categories of micro goals across episodes.

Table 25. Frequency of major categories of episode goals underpinning episodes.

Major categories of episode goals		**Episodes (n = 137)**
Single episode goal	Content focused goal	65 (47%)
	Student focused goal	35 (26%)
Dual episode goals	1 Content and 1 Student focused goal	24 (18%)
	2 Content focused goals	9 (7%)
	2 Student focused goals	4 (3%)

The majority of the preservice teachers' episodes were underpinned by a single episode goal which tended to be one that focused on content (47% of episodes). When dual episode goals underpinned episodes, the combination of content and student focused goals occurred the most often. Of the 24 episodes shown in the table where the preservice teachers enacted MCK to achieve dual content and student focused goals, almost half took place because the preservice teachers wanted first, to address a particular point of confusion (student focused goal) and simultaneously, to explicitly highlight certain mathematical content (content focused goal). Particular combinations of content and student focused goals resulted in the participants teaching different types of MCK, a finding which is elaborated in chapter 5.

4.6.1.2 Reading across episodes: Influences impacting choice of goals

In the reflections of 106 episodes, the preservice teachers referred to influencing elements that contributed to their choice of one or two episode goal(s) and/or their consideration of discarded goal(s). Almost all those elements were described earlier in this chapter. One extra influencing element, the preservice teachers' university studies, was mentioned once by two participants but was not considered a significant influence on MCK related decision making across multiple lessons or participants.

Reading across the episodes, several combinations of influences were implicated in the preservice teachers' explanations for their choice of single or dual micro goal(s). The combinations of influences described in this section refer only to compatible sets of influences that contributed to the participants' choice of goal. Table 26 reports the influences on micro goals by episode because it was not possible to ascertain if particular influences impacted only one goal, when dual goals were noted for an episode. Influence 2 has not been included in the table because the macro and meso goals were not referred to by the participants in individual episode reflections.

Table 26 shows that classroom circumstances that catch the attention of preservice teachers and the judgements they make about students heavily influence the micro goals they decide to pursue. Classroom circumstances and/or student judgements were implicated in 87% of the 106 episode reflections in the table. When rationalising their choice of goals, the preservice teachers referred to their judgements about students and how they learn mathematics (reflections of 46% of episodes) more often than to their own MCK (reflections of 16% of episodes). This was despite forming goals that focused on mathematical content more often than student focused goals (see Table 24). This finding suggests that the preservice teachers' judgements about students can lead them to form content or student focused goals. Classroom circumstances can also lead to content and student focused goals and chapter 5 reveals the significance of these combinations where enacted MCK is concerned. The absence of explanations regarding the participants' choice of goals in 31 episodes were also examined for any trends regarding the MCK that manifested in the episodes, but no patterns were evident.

Table 26. Preservice teachers' self-reported influences on their choice of micro goals

No. of influences	Influence 1 (Practicum context)	Influence 3 (Class circumstances)	Influence 4 (MCK)	Influence 5 (Student judgements)	University studies	Episodes (n = 137)
0						31 (23%)
1				✓		28 (20%)
		✓				22 (16%)
			✓			8 (6%)
	✓					4 (3%)
					✓	1 (1%)
2		✓		✓		21 (15%)
			✓	✓		5 (4%)
	✓			✓		4 (3%)
		✓	✓			3 (2%)
	✓	✓				2 (1%)
	✓		✓			1 (1%)
3		✓	✓	✓		2 (1%)
	✓	✓	✓			2 (1%)
	✓	✓		✓		1 (1%)
	✓			✓	✓	1 (1%)
4	✓	✓	✓	✓		1 (1%)
Total	16 (12%)	54 (39%)	22 (16%)	63 (46%)	2 (1%)	137 (100%)

4.6.1.3 Reading across episodes: Influences impacting choice of MCK

In addition to identifying influences on the goals they formulated, the preservice teachers also identified influences on their choice of particular MCK to deliver in pursuit of those goals. They did so in their reflections of 97 episodes. Similar to the analysis of the influences impacting the preservice teachers' choice of goals, the researcher read across the episodes to identify which influences or combinations of influences were prevalent when preservice teachers reflected on choosing particular MCK to present during instruction to achieve their goals. The results of the analysis are presented in Table 27. The preservice teachers' own schooling experiences and their university studies each feature once in the table. As the presence of each element was limited to only one comment by one preservice teacher and the elements themselves did not form part of the major influences described previously in this chapter, both elements have been grouped together in the table and the pertinent element identified within the cells of the table.

A stronger presence of the preservice teachers' own MCK is evident in Table 27. Perhaps not surprisingly, the preservice teachers regularly draw on their own MCK when they made choices about which aspects of their MCK they should teach during their lessons. However, they also relied quite heavily on their judgements about students to inform their choice of enacted MCK and when they did so, they tended to present certain types of MCK. This finding is explored in detail in chapter 5. The influence of MCK and judgements about students on the preservice teachers' choice of MCK show how decisions to enact MCK during instruction necessarily involve pedagogical and mathematical considerations. Aspects of the preservice teachers' pedagogical content knowledge, which manifested in this study in the judgements they made about students, and their own MCK, regardless of their sturdiness, feature strongly in the preservice teachers' thoughts about presenting MCK within a live mathematics lesson.

Table 27 shows that the preservice teachers referred to a single influence in just over half (51%) of the episodes, when they explained their choice of MCK to enact. For the remainder of the explanations they gave that implicated influences, however, sets of compatible influences were indicated. An example from Sam's interview illustrates how he drew on three different influences when reflecting on the MCK he chose to enact in one of his episodes.

Table 27. Preservice teachers' self-reported influences on their choice of MCK to enact

No. of influences	**Influence 1 (Practicum context)**	**Influence 3 (Class circumstances)**	**Influence 4 (MCK)**	**Influence 5 (Student judgements)**	**University studies or own schooling**	**Episodes (n = 137)**
0						40 (29%)
1			✓			39 (28%)
				✓		29 (21%)
	✓					2 (1%)
2			✓	✓		19 (14%)
	✓			✓		2 (1%)
	✓		✓			3 (2%)
	✓				University studies	1 (1%)
3	✓			✓	Own schooling	1 (1%)
	✓		✓	✓		1 (1%)
Total	9 (7%)	0 (0%)	62 (45%)	52 (38%)	2 (1%)	137 (100%)

Sam chose at one point in his first lesson to present the equation, $5y = 45$, from the class textbook for his students to solve but modelled a solution using the transposing method, rather than the balance method presented in the textbook. His presentation of the question and the solution was defined as an episode, underpinned by a micro goal to develop his students' knowledge of a procedure. The rationale he provided for presenting the particular question and solution path in the episode, as a means to achieve his episode goal, reflects the influence of the practicum context, his own MCK, and his judgements about how students should learn mathematics.

Sam explained that the question he presented was taken directly from the textbook, reflecting the influence of the practicum context. Sam's justification for presenting a solution path involving the transposing method implicated two more influences. First, Sam explained how he had prioritised his own preferred method for solving equations (the transposing method) over the textbook method (the balance method), reflecting the influence of his own MCK preference. He commented, "I prefer, well, I guess the quickest way is just to take it over and swap the sign. So that would be my preferred way." Second, Sam justified his choice of solution path as being a better option for students, as he reflected on the textbook method, "Some of the ways they [the textbook] do things I don't like so much but you can just change them to how you want the kids to do it…They [the students] don't like this [balance method] at all." Sam's judgements about the methods that best suit students, his own MCK, and elements of the practicum content influenced the MCK he enacted, illustrating how multiple influences can collectively impact preservice teachers' MCK related decisions.

Sam's story also suggests that influences on MCK related decisions do not always lead preservice teachers in the same pedagogical direction. Although Sam chose in this instance to draw in part on three different influences to inform his decision, he could have chosen to forego his own preference in favour of the textbook method. In some circumstances, the preservice teachers were prepared to discard an influence altogether if conflicts existed between the sets of influences they managed.

4.6.1.4 Reading across episodes: Competing influences

The analysis of the preservice teachers' explanations for either forming a goal or enacting particular MCK to pursue a goal revealed a number of competing influences. Across the reflections provided for the episodes, the preservice teachers noted 60 conflicts between influences that they resolved by retaining one influence and discarding another. Figure 7 shows the combinations of influences that competed for the preservice teachers' approval and the influence that was prioritised in different circumstances. The presence of each set of competing influences is presented as a percentage of the 60 conflicts experienced by the preservice teachers. Subcategories are also provided to demonstrate the various conflicts that the preservice teachers managed.

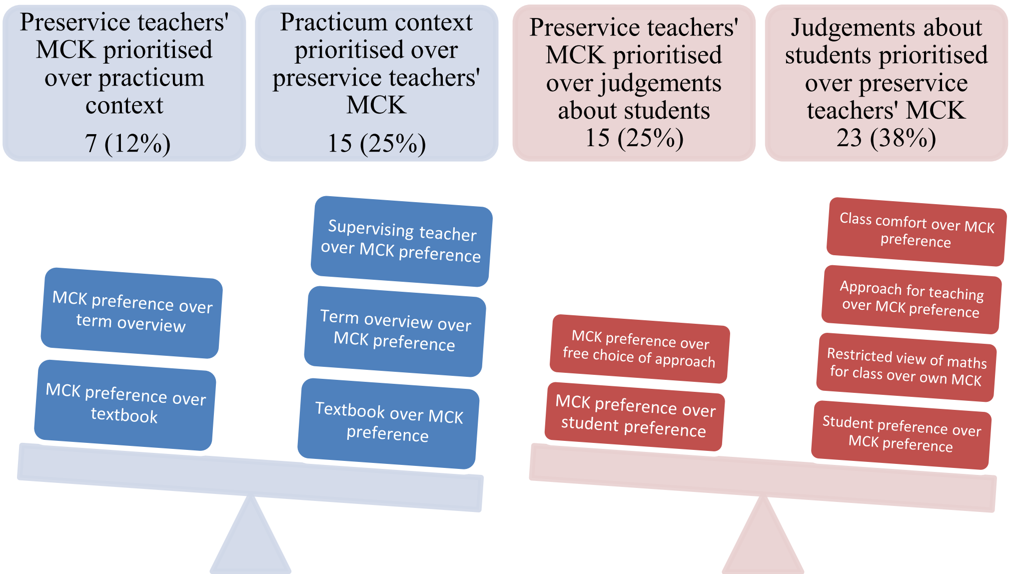


Figure 7. Competing influences articulated by the preservice teachers (n = 60)

Figure 7 shows that preservice teachers are more likely to resolve conflicting influences involving their mathematical preferences by prioritising influences other than their own MCK. The preservice teachers' practicum responsibilities and student judgements were privileged when competing influences were experienced by the preservice teachers.

Together, the micro goals and explanations offered by the preservice teachers in their episode reflections highlight the number of influences which preservice teachers are aware of, when deciding on mathematical content to present in a live classroom context.

4.6.2 Combinations of influencing elements implicit in the data

In contrast to section 4.6.1, this section presents three combinations of influencing elements that were inferred by the researcher. The combinations reported are those that were found to have a close association with a particular type and/or quality of enacted or intentionally withheld MCK.

The first combination of interest comprised two influencing elements in the category, “classroom circumstances”. Those elements were “classroom events” and “instructional setting”. Figure 8 presents the 137 episodes grouped according to (a) the presence of a classroom event that prompted an episode and (b) the instructional setting in which the episode took place.

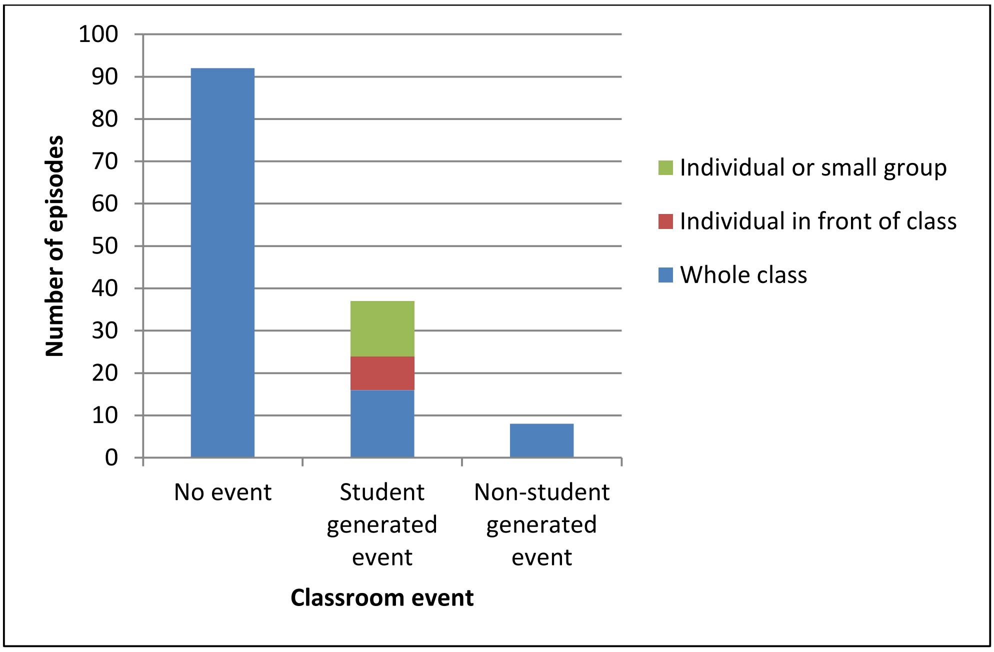


Figure 8. Distribution of episodes (n = 137), by classroom event and instructional setting

Decisions not prompted by an event were described in the introduction of this chapter as preactive decisions. Figure 8 shows that two thirds of all episodes reported in this study involved preactive decisions made by preservice teachers while teaching in a whole of class instructional setting. Given the preservice teachers were able to plan the content delivery in episodes based on preactive decisions, those episodes might be expected to feature stronger MCK than episodes where the participants had only seconds to make their decisions. However, as chapter 5 will reveal, this was not the case. In contrast to the preactive decisions, the preservice teachers made interactive decisions regarding MCK, prompted by a student generated event, in instructional settings involving small or large groups of students. The combination of a student prompted classroom event and a small instructional setting was a particularly significant influence on enacted MCK. This finding is elaborated in chapter 5.

An analysis of the episodes associated with particular lesson phases and instructional settings brought to light important combinations of influencing elements that collectively impacted MCK related decisions. Figure 9 provides an overview of all episodes, grouped according to the corresponding lesson phase and instructional setting.

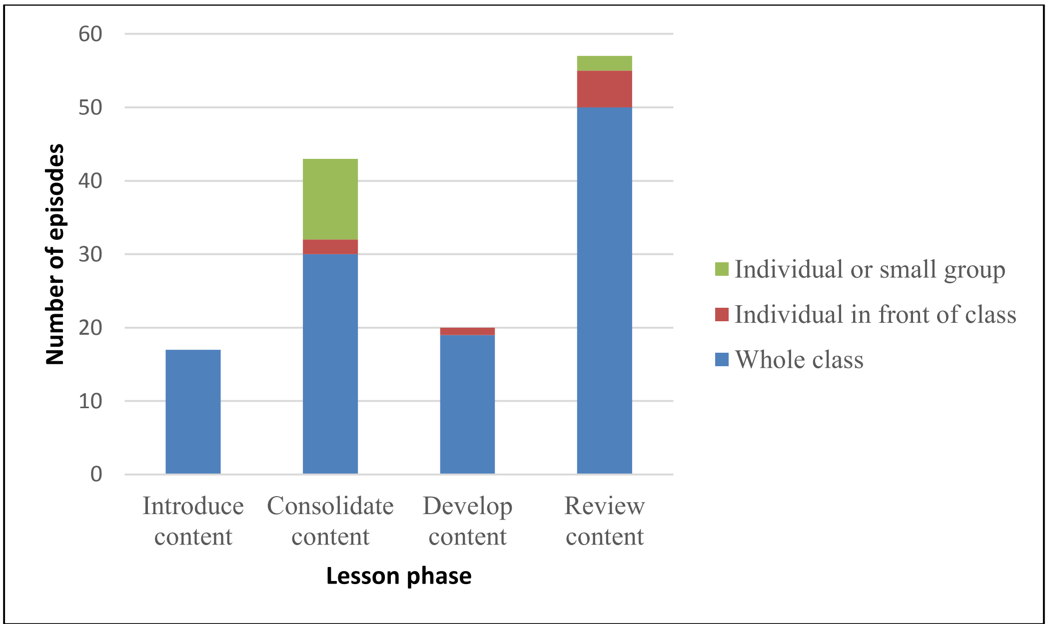


Figure 9. Distribution of episodes (n = 137), by lesson phase and instructional setting

Figure 9 shows that lesson phases where the preservice teachers aimed to consolidate their students' knowledge of content featured a significant number of episodes that took place with small numbers of students. In contrast, the introduction phase featured no episodes of this type. How the MCK that the preservice teachers chose to deliver varied with respect to these two element combinations is discussed in chapter 5, when the enacted MCK for the corresponding episodes is examined.

On 25 occasions, a preservice teacher noted in his or her interview that he or she decided to intentionally withhold particular MCK from an episode. Using the participants' reflections of those decisions and the lesson footage, the researcher was able to ascertain the associated instructional setting, the presence or absence of a classroom event that prompted the decisions, and the lesson phase in which the decisions were located. Figure

10 illustrates the combinations of those three influencing elements (instructional setting, classroom event, and lesson phase) that resulted in MCK being withheld.

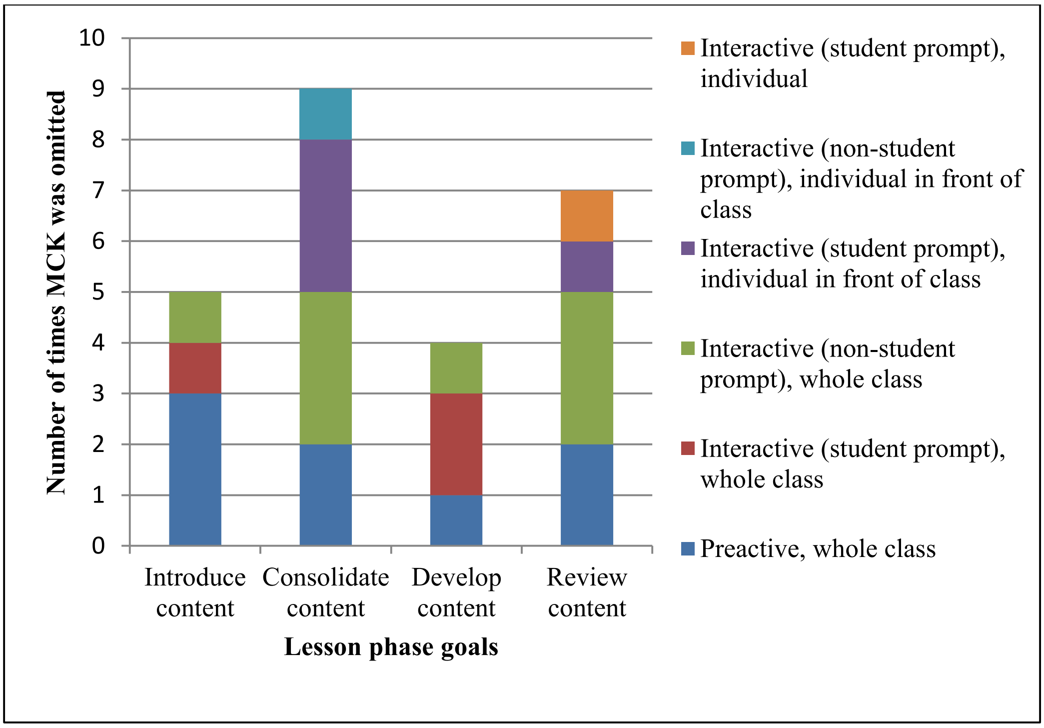


Figure 10. Distribution of moments within a lesson when MCK was omitted (n = 25), by lesson phase, classroom event, and instructional setting

The graph shows that preservice teachers do make some decisions to intentionally hold back particular MCK in the planning stages of their lessons and those decisions are reinforced when the preservice teachers implement their lesson images. As lessons unfold, the preservice teachers make further decisions not to enact certain aspects of their MCK. Both of these circumstances tend to occur when the preservice teacher interacts with one or more students in front of the whole class. This trend is revisited in chapter 5, when the thoughts behind the preservice teachers' decisions to omit particular MCK are examined.

4.7 Conclusion

The findings of this chapter indicate the complexity of the MCK related decision making process for preservice teachers. Elements of their practicum context (influence 1), their lesson and lesson phase goals (influence 2), and different circumstances that occur within the lesson (influence 3) are weighed up by the preservice teachers, alongside their own MCK (influence 4) and the judgements they hold about students (influence 5). These influences collectively inform their decisions concerning the choice of micro goals (influence 2) and the mathematical content that they decide to enact or withhold as they pursue the goals that they retain.

While the presence of multiple influences is often a harmonious one, competing influences can create conflicts which preservice teachers must resolve by prioritising one influence over another. The management of numerous decisions, involving multiple influences and often unfamiliar pedagogical territory, makes the task of teaching algebra a challenging and demanding experience for preservice teachers. Chapter 5 provides a description of the qualitative differences found in the MCK that the preservice teachers enacted in their algebra lessons. The differences are examined in light of the decision making influences that have been reported in this chapter.

Chapter 5: Research findings 2: The MCK that preservice teachers enact

5.0 Introduction

The research questions that directed the investigation of secondary preservice teachers' mathematical content knowledge (MCK) in this study were as follows:

1. What elements influence the decisions secondary preservice teachers make regarding the mathematical content knowledge (MCK) they enact when teaching lower secondary algebra?

2. What is the mathematical content knowledge (MCK) that secondary preservice teachers enact when teaching lower secondary algebra?

In chapter 4, the first research question was partially addressed by presenting the findings that identified the influences on preservice teachers' MCK related decisions. In this chapter, the MCK that preservice teachers subsequently delivered from making those decisions is investigated. Enacted MCK is reported in this chapter in terms of the three types of MCK that the researcher discerned from the preservice teachers' instructional actions during their algebra lessons: knowledge of algebraic ways of thinking (AWOTS), conceptual knowledge, and procedural knowledge. The chapter also draws together the two sets of findings, namely, the influences on MCK related decision-making and the nature of the MCK enacted, to identify trends between the presence of each MCK type and the influencing elements described in the previous chapter.

The researcher analysed the video footage of each episode to categorise the enacted MCK and to make connections between that MCK and the influences that shaped the decisions of each episode. Interview data, as reported in chapter 4, provided frequent insights into what influenced the preservice teachers' decisions regarding the MCK they chose to enact – or not to enact. In a more limited fashion, the interviews also provided insights into preservice teacher MCK that was not present in the episodes. While no additional AWOTS manifested in the interviews, the preservice teachers did have additional pertinent knowledge concerning conceptual and procedural knowledge.

The three categories of MCK evident in preservice teacher instruction - knowledge of AWOTS, conceptual knowledge, and procedural knowledge - appeared in different proportions and combinations. Figure 11 provides an overview of the three MCK types presented by the preservice teachers, showing the combinations that manifested in the episodes. No episodes exhibited AWOTS in isolation but either procedural or conceptual knowledge figured alone in 64 (47%) of the 137 episodes. For the remaining 73 episodes, combinations of two or more MCK types were present. The preservice teachers delivered procedural knowledge the most often during instruction. This knowledge type was evidenced either in isolation or with another type of MCK in 91% of all episodes. In contrast, AWOTS were taught by the preservice teachers in slightly less than half of the episodes (49% of episodes) and conceptual knowledge manifested in just over one third of episodes (35% of episodes).

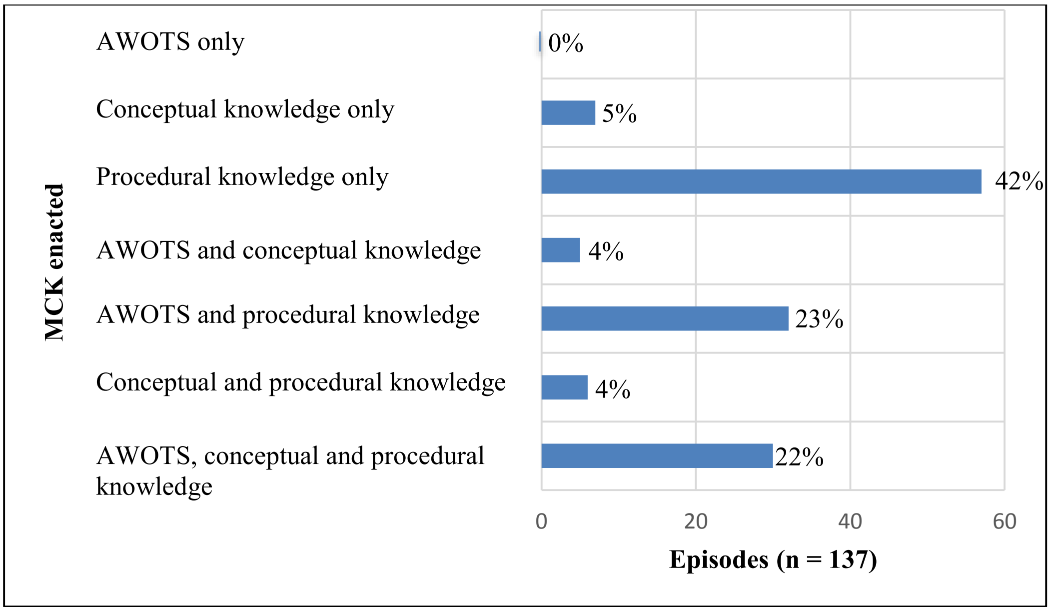


Figure 11. Presence of MCK types in teaching episodes

The graph in Figure 11 reflects the high occurrence of procedural knowledge and reveals that in 57 episodes (42% of all episodes), procedural knowledge was presented in isolation. Given that substantial conceptual knowledge and AWOTS would be considered positive additions when teaching students procedures (Driscoll, 1999; Hiebert & Le Fevre, 1986; Kilpatrick et al., 2001; Skemp, 1979), it would appear that, overall, conceptual knowledge and AWOTS were not enacted often enough by the preservice teachers.

While the summary and the figure above provide a useful 'birds-eye view' of the MCK delivered by the preservice teachers across their ten lessons, they also "force data into shapes that are superficially comparable" (Miles et al., 2014, p. 102). The statistics reported above do not take into account the quality of the MCK enacted or the influences on the preservice teachers' decisions to explicitly communicate each type of MCK. Thus, the major sections of this chapter provide an analysis of the three types of MCK taught by the participants and the influences that led to the enactment of each type. To reiterate, those influences and the elements comprising each influence are summarised in Table 28 below.

Table 28. Influencing elements impacting preservice teachers' MCK related decisions

Influence	Influencing elements
Practicum context	Supervising teacher Term overview Perceptions of the mathematical ability of the class Class textbook
Preservice teachers' goals	Macro level goals (lesson goals) Meso level goals (lesson phase goals) Micro level goals (episode and discarded goals)
Live classroom circumstances	Classroom events Instructional setting
Preservice teachers' MCK	Common content knowledge Specialised content knowledge Horizon knowledge
Preservice teachers' judgements about students	How students learn mathematics Students' mathematical needs Content to which students should be exposed

The preservice teachers' enactment of their knowledge of AWOTS is discussed in section 5.1, followed by their enactment of conceptual knowledge in section 5.2 and procedural knowledge in section 5.3. In each chapter section, the MCK itself is examined in three ways. First, the type of MCK present in the episodes is organised into categories derived from the literature related to (a) AWOTS (Cuoco et al., 2010; Driscoll, 1999; Harel, 2008c; Harel et al., 2008), (b) conceptual knowledge (Davis, 2008a; Even, 1993; Skemp, 1976; Tall & Vinner, 1981), and (c) procedural knowledge (Greeno, 1978; Schneider et al., 2011; Star, 2005). Second, examples of specialised MCK that the participants presented are described. Third, limitations of the MCK for teaching purposes are reported, including pertinent MCK that was noticeably absent, whether intentionally or not. The qualitative judgements made by the researcher about the MCK delivered and its suitability for secondary teaching were guided by the literature on mathematical knowledge needed for the work of teaching algebra (e.g., Ball et al., 2008; Ma, 1999; McCrory et al., 2012).

The description of MCK includes a discussion of influencing elements that appeared to impact the preservice teachers' decisions pertaining to the type of MCK enacted. In each chapter section, results of variable-oriented analyses, described in chapter 3, are provided, revealing the influencing elements that were most prevalent in teaching episodes when certain types and quality of MCK were present. Trends are reported for episode goals when a particular goal type was present in at least 50% of episodes featuring a particular MCK type. Identified patterns are presented with quantitative summaries and supported with qualitative descriptions and vignettes to illustrate and explore potential areas of significance for preservice teachers' MCK related decisions and teaching actions.

5.1 Preservice teachers enact algebraic ways of thinking

Why, why, why was I doing it all this time if not to get to an answer?

(Sam reflects on confirming a student's hunch about the purpose of a procedure)

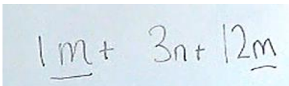
5.1.1 The algebraic ways of thinking enacted

All preservice teachers explicitly enacted knowledge of AWOTS but they rarely paid them attention beyond a cursory reference. The AWOTS identified in the literature and described in chapter 2 are as follows:

- Manipulating with purpose (Cuoco et al., 2010; Harel, 2008c; Harel et al., 2008);
- Doing-undoing (Driscoll, 1999; Kieran, 1992);
- Algebraic invariance (Cuoco et al., 2010; Harel, 2008c);
- Building rules to represent functions (Capraro & Joffrion, 2006; Driscoll, 1999);
- Abstracting from computation (Driscoll, 1999; Pjanić & Nesimović, 2013).

Evidence of enacted MCK, as defined in this study, requires knowledge to take an observable form in the classroom. Hence, only explicit references to AWOTS were deemed evidence of enacted ways of thinking. Without an explicit reference, it was not possible to ascertain if the preservice teachers held more than a tacit knowledge of an algebraic way of thinking. The researcher discerned an implicit knowledge of AWOTS from the video footage in almost every episode but discerned one or more explicit references to the ways of thinking in 67 of the 137 episodes (49% of all episodes), across the ten lessons. Table 29 provides an overview of the types of AWOTS that the preservice teachers chose to present, the frequency with which they were evidenced, and a short illustration of how the preservice teachers typically communicated each way of thinking in their lessons. The total percentage of episodes exceeds 100% because more than one algebraic way of thinking featured in some episodes.

Table 29. AWOTS evidenced in episodes

Algebraic way of thinking	**Episodes (n = 67)**	**Episode excerpt**
Manipulating with purpose	35 (52%)	Ben: What do we wanna do to the x? Student: Isolate it Ben: Isolate it…So you wanna get it on its own.
Doing-undoing	24 (36%)	"To get rid of the times, you have to divide." (Grace, as she isolates x in the equation, $2x = 2 - 3y$)
Algebraic invariance	20 (30%)	"This m here, you can write it as one m if you want." (Thomas, adding the "1" to the front of the first term below) 
Building rules from functions	11 (16%)	"If we say the number I was thinking in my head was 'a', and I add six to it, the answer is thirteen…So I'm talking an English sentence. We need to work out what it is in a mathematical sentence." (William)
Abstracting from computation	0 (0%)	None available

The high proportion of "manipulating with purpose" and "doing-undoing" ways of thinking was due to the preservice teachers' references to them when they isolated a pronumeral in a linear equation. The preservice teachers offered mathematical purposes for procedural steps and referred to undoing operations performed on a pronumeral as they manipulated linear equations in class. These two ways of thinking were explicitly articulated either in isolation or together by the preservice teachers in 44 episodes, i.e., almost two thirds of the 67 episodes featuring AWOTS. The preservice teachers spoke less often of changing the form of an algebraic term, expression, or equation without changing its value (algebraic invariance) and translating a worded scenario into an equation (building rules from functions). No preservice teacher made an explicit reference to the abstracting from computation way of thinking in their lessons.

Lengthy discussions of AWOTS rarely formed part of the findings above. Only William devoted a significant amount of class time to the discussion of an algebraic way of thinking and his attempt to explicitly highlight the doing-undoing way of thinking is discussed later in this section.

The preservice teachers very occasionally showed glimpses of a more specialised knowledge of AWOTS that is needed for the work of teaching. In only 11 episodes (8% of all episodes and 16% of the episodes featuring AWOTS) did the preservice teachers demonstrate an unpacked and connected knowledge of an algebraic way of thinking, two key aspects of specialised algebraic knowledge for teachers (McCrory et al., 2012). The beginnings of specialist knowledge of AWOTS were evidenced in this study when preservice teachers made a point of explicitly highlighting connections between AWOTS and related procedural steps or concepts with their students.

Specialist knowledge of an algebraic way of thinking was shared by all preservice teachers during six of the ten observed lessons. When an episode was coded in this study as featuring a specialised knowledge of a way of thinking, that way of thinking was not only explicitly communicated by a preservice teacher but explicitly connected to a mathematical procedure and/or concept. In Sam's lesson, for example, he took care to attend to a new feature of an example given to his class and the purpose behind a particular procedural step. Presenting the equation $6 - x = x$ to his students, he began by highlighting a feature of the example new to his students then made a connection between that feature, the "manipulating with purpose" way of thinking, and a procedural step, telling his students,

> So this one is a little bit different because we have an x on both sides of the equation. We want to get x on one side of the equation because that's what we're used to seeing. We've got an x here and an x there, [pointing to each side of the equation] so what I would do is get this one over to the other side. We want to take this x [pointing to the x in the left hand expression] over to the other side [points to the x in the right hand expression].

In Sam's explanation to his class, he highlighted an explicit connection between the features of the example, the "manipulating with purpose" way of thinking, and certain algebraic manipulations that he explained had a particular mathematical purpose. Limitations of Sam's explanation were also evident and are discussed in section 5.1.3 but his actions demonstrate, as the actions of the other five preservice teachers also did, a specialised knowledge of some facets of AWOTS.

5.1.2 Influences associated with algebraic ways of thinking

The thoughts behind the 67 episodes provide insights into the elements and combinations of elements that appear to lead preservice teachers to enact AWOTS. The influencing elements that were associated with the 67 episodes featuring any algebraic way of thinking were analysed by the researcher. The analysis revealed four sets of influencing elements that tended to result in AWOTS. They are as follows:

- Episode goals that emphasise content beyond mastery of procedures or that address student confusion;
- Small instructional settings involving conversations with confused students;
- Lesson phases in which content is introduced as a result of preactive decisions;
- Interactive decisions leading participants to unpack their MCK during instruction.

The four sets of influences are elaborated and illustrated below, using quantitative summaries and vignettes.

5.1.2.1 Episode goals leading to algebraic ways of thinking

Preservice teachers did not always pay careful attention to the mathematical ideas inherent in a procedure but when they did, AWOTS tended to manifest. The preservice teachers provided 88 episode goals for the 67 episodes where AWOTS were present. Table 30 shows the distribution of those 88 goals according to the nine episode goal categories described in chapter 4 and compares their presence with all episode goals in that category.

Table 30. Type and relative frequency of episode goals for AWOT episodes

Major category of episode goals	**Type of episode goal**	**Total goals for type (n = 174)**	**Goals for AWOTS**		**Goals for *specialised* knowledge of AWOTS**	
			No. (n = 88)	**Percentage of total goals by type**	**No. (n = 14)**	**Percentage of goals for AWOTS by type**
Content focused goals	Develop students' knowledge of procedures	60	22	37%	3	14%
	Teach students appropriate use of mathematical language	16	10	63%	1	10%
	Connect procedure with a concept	12	8	67%	2	25%
	Associate procedure with certain types of solutions	10	2	20%	0	0%
	Connect procedure with mathematical purpose	9	9	100%	3	33%
Student focused goals	Address student confusion	32	23	72%	3	13%
	Value and/or encourage student contribution	12	4	33%	1	25%
	Gauge student knowledge	12	4	33%	0	0%
	Avoid student confusion	11	6	55%	1	17%

Not surprisingly, Table 30 shows that when the preservice teachers decided to highlight the mathematical purpose of an algebraic procedure, they taught AWOTS each time. Furthermore, the table shows that the preservice teachers also tended to present AWOTS when they aimed to (a) connect a procedure with a concept or (b) to teach students about appropriate ways of communicating mathematics. The goals in these three content focused categories were established by all six preservice teachers in nine out of the ten observed lessons. Only Thomas' first lesson did not feature any explicit references to ways of thinking.

William was the preservice teacher whose intents and teaching actions focused most explicitly on an algebraic way of thinking. In five episodes during his lesson, William taught an algebraic way of thinking as he undertook a number of teaching actions to highlight the doing-undoing way of thinking to his students. One of the episodes, which is presented in Appendix O, shows William attempting to make an explicit connection between a procedural step, performing the 'opposite' operation, and the algebraic way of thinking behind that step, to 'undo' the operation performed on the pronumeral.

In his reflection, William drew on his MCK twice when he articulated goals and rationales for his actions in the episode. First, he provided a content focused episode goal, noting a specific desire to connect an aspect of a procedure with its mathematical purpose (see Appendix O, episode reflection 1). Second, he provided a mathematical rationale for having probed particular students' ideas and having prioritised certain student responses over others, in pursuit of his intended goal (Appendix O, episode reflection 2). In the episode itself, William's knowledge of an algebraic way of thinking manifested in his teaching actions, including explicit verbal references to "doing the opposite" and "working backwards". While William's attempt was not perfect, his awareness of an algebraic way of thinking inherent in the procedure and his intent to share that aspect of mathematical content with his students led to a mathematically stronger episode than one that only focused on procedures and rules alone. William's episode and reflection shows how preservice teachers choose to share AWOTS when they pay attention to mathematical ideas that relate to the algebraic procedures they are teaching.

The attention the participants paid to their students' mathematical troubles also produced positive results, where AWOTS were concerned. Almost three quarters of preservice teachers' intentions to address student confusion led to teaching actions featuring

AWOTS (see Table 30). For example, Grace explicitly referred to AWOTS in one episode when she helped a student solve a set of simultaneous equations. In her interview, Grace reflected, “I said to him [the student], ‘Remember what we are trying to find?’ And he’s like, ‘I don’t know.’ So said to him, ‘Remember we’re trying to find y.’” Addressing student confusion was an episode goal that was established at some point by every preservice teacher in every lesson when explicitly enacting a way of thinking. This finding suggests a close association between thoughts of helping confused students and actions involving AWOTS.

The prevalence of certain episode goals leading to AWOT episodes was not reflected in the goal types underpinning the subset of episodes featuring specialised knowledge of AWOTS. Table 30 (far right columns) shows that no goal type tended to lead to specialised knowledge of AWOT episodes significantly more than others. The absence of a notable goal type for this kind of specialised MCK is in contrast to the findings reported later in this chapter which identify goal types that tend to lead to specialist conceptual and procedural knowledge.

5.1.2.2. Small instructional settings involving conversations with confused students

Opportunities to teach AWOTS existed in all instructional settings but it was within smaller settings where the participants generally took up those opportunities. Table 31 shows the type and relative frequency of instructional settings for episodes featuring any AWOTS and specialised forms of that knowledge.

Table 31. AWOT episodes by instructional setting

Type of instructional setting	**Total episodes for type**	**AWOT episodes**		***Specialised* knowledge of AWOT episodes**	
		No.	**Percentage of total episodes for type**	**No.**	**Percentage of AWOT episodes**
Whole class	116	53	46%	8	15%
Small group or individual	13	9	69%	2	22%
One student, conducted in front of class	8	5	63%	1	20%
Total	137	67	49%	11	8%

Table 31 shows no instructional settings that were closely associated with episodes featuring specialised knowledge of AWOTS. However, AWOTS of any kind featured in over two thirds of all episodes that took place in a small instructional setting. Of those nine episodes, highlighted in the table, a goal to address student confusion underpinned eight of them. This finding suggests that a private discussion (i.e., without the remainder of the class listening) with students who are confused about mathematical content is an environment that is conducive to preservice teachers presenting AWOTS.

5.1.2.3 Lesson phases in which content is introduced as a result of preactive decisions

Planning to introduce new content was associated with enacted AWOTS. Episodes with AWOTS were grouped according to the lesson phases in which they were nested and the presence or absence of a classroom event that prompted the decisions in each episode. Those episodes were then compared with the total number of episodes in each lesson phase. The results are presented in Table 32.

Table 32. AWOT episodes by lesson phase and classroom event

Type of lesson phase	**Total episodes for type**	**AWOT episodes**		**AWOT episodes by classroom event**	
		No.	**Percentage of total episodes for type**	**No event**	**Classroom event**
Introduce content	17	12	71%	9	3
Consolidate content	43	18	42%	3	15
Develop content	20	11	55%	9	2
Review content	57	26	46%	17	9
Total	137	67	49%	38	29

The table reveals that in the majority of episodes (71%) located within lesson phases aimed at introducing mathematical content, AWOTS were present. The most common algebraic way of thinking taught during the "introduce content" lesson phases was "building rules from functions" as preservice teachers modelled how to represent a worded problem with an equation. Fewer instances of "manipulating with purpose" and “doing-undoing” were present when preservice teachers spoke about manipulating equations to isolate a pronumeral.

The table also shows that three quarters of episodes featuring AWOTS in introduction phases were the result of preactive decisions (i.e., no classroom event), suggesting that preservice teachers may plan to enact mathematical purpose alongside new procedures. However, the preservice teachers in this study did not indicate that they were aware of giving more attention to ways of thinking in lesson phases where they were introducing content. It would seem that preservice teachers, perhaps unknowingly, plan to take more care highlighting the mathematical purposes behind new procedural steps and may assume that the mathematical purpose is already known when they plan to consolidate familiar content. This may explain why AWOTS were more often the result of interactive decisions than preactive decisions when lesson content was mediated in the consolidate content lesson phases.

5.1.2.4 Interactive decisions leading participants to unpack their MCK during instruction

The preservice teachers’ students inadvertently directed their teachers to deliver AWOTS. Fifteen episodes with AWOTS were the product of interactive decisions in “consolidate content” lesson phases, compared with only three resulting from preactive decisions (see Table 32). Interactive decisions also led to AWOTS in 14 more episodes, across the other three phases and nine of those were in the review content phase where familiar content was once more being presented. The majority of interactive decisions leading to AWOTS (24 out of the 29 episodes) were prompted by a classroom event generated by a student question or comment. The impact of student prompted classroom events was particularly significant because for at least four of the preservice teachers, the events prompted certain AWOTS to come to the fore that might otherwise have laid dormant.

AWOTS seemed so obvious to the preservice teachers at times that they appeared baffled at having to explicitly address them during instruction. Four of the preservice teachers, Sam, Ben, Grace, and Kate, commented on their surprise at having to be so obvious about the purpose behind their mathematical actions. Sam, for example, carefully explained for several minutes in class how to solve the equation, $2x + 7 = 17$, using the backtracking method and his boardwork is shown in Figure 12.

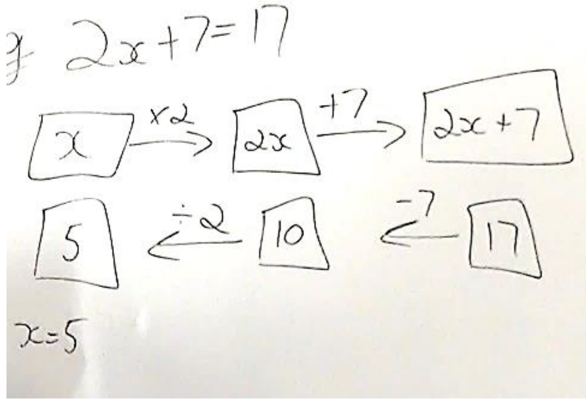


Figure 12. Sam carefully steps out the backtracking method

After his explanation, one of his students asked, “Is that how we get the answer, Sir?” Sam confirmed his student’s rather confused hunch, highlighting the mathematical purpose of the procedure only after having been prompted to do so by the confused student’s question. As Sam watched the video footage in his interview, his laughter and comments, “Well I thought, ‘Why, why, why was I doing it all this time if not to get to an answer!’”, reflected his surprise at having to spell out the point of the procedure. Similar stories of surprise and disbelief were shared by Ben, Grace, and Kate, yet no preservice teachers showed surprise at needing to enact conceptual or procedural knowledge at any point in their lessons. The surprise shown by the majority of preservice teachers only for this facet of MCK creates a curious trend where AWOTS are concerned.

The preservice teachers’ realisations during instruction to teach AWOTS appeared to be influenced in two ways. First, the student prompt that led to an interactive decision appeared to trigger a change in the pedagogical tack the preservice teachers were currently taking, highlighted by their students’ real mathematical needs. Second, the change in instructional direction, prompted by the students, required the preservice teachers to re-examine the content they had presented and to access specific AWOTS to clarify their students’ points of confusion. The implicit presence of AWOTS in the episodes leading up to the preservice teachers’ surprise at having to explicitly articulate them suggests that the form in which they held their knowledge of AWOTS may have been initially too compressed to access. Only when interactions with students in the moment required them to rethink the mathematics content they were presenting, did they become more consciously aware of the purpose behind certain manipulations and share that knowledge in class. It is likely that the preservice teachers' pedagogical content knowledge (PCK) concerning the content that students need, would also have developed as a result of the same interactions. However, the preservice teachers' MCK still needed to decompress for this to occur. The classroom events generated by students appeared to impact the form of the preservice teachers' MCK and refine their judgements about what students need which, in turn, led to them teaching AWOTS that they had not previously thought to share with their students.

5.1.3 Limitations in the algebraic ways of thinking enacted

Preservice teachers' instructional actions concerning AWOTS were limited by what was enacted and what was not. The main limitations discernible in the preservice teachers' actions in the classroom were a lack of adequate mathematical depth and verbal clarity when delivering AWOTS. The preservice teachers also missed numerous opportunities to teach AWOTS in their episodes. For example, despite its presence in 20 episodes, the "algebraic invariance" way of thinking was particularly notable in its absence from many other episodes.

Emergent specialist knowledge would most aptly describe the AWOTS that were present in the preservice teachers' instructional actions. In section 5.1.1, Sam's attempt to share his knowledge of an algebraic way of thinking reflects many of the preservice teachers' actions, where a way of thinking was mentioned or briefly referenced but their explanation was far from comprehensive. Sam's clumsy justification for manipulating an equation, "because that's what we're used to seeing," demonstrated a lack of explicit knowledge about why it was mathematically advantageous for his students to have x on only one side of the equation. This suggests that while he had some connected knowledge of equation structure, algebraic manipulations, and the "algebraic invariance" and "manipulating with purpose" ways of thinking, Sam's specialist knowledge was at an early stage of development. The surface was scratched, where AWOTS were concerned, but not deeply enough by Sam or any other participant.

The preservice teachers' use of imprecise mathematical language weakened the mathematical quality of many episodes featuring AWOTS. An examination of the language, both verbal and written, that preservice teachers used yielded almost no significant written inaccuracies. However, the preservice teachers' classroom talk revealed five categories of verbal imprecision referred to in the literature that negatively impacted the clarity of mathematical ideas presented. Those categories which were described in chapter 3 are provided below.

- Language is used in an incorrect mathematical context (Heaton, 1992; Hill, Blunk et al., 2008).
- Language features non-mathematical, informal code (Dunn, 2004; Falle, 2005; Hill, Blunk et al., 2008; Zazkis, 2000).
- Language is ambiguous (Sleep & Eskelson, 2012; Smith, 1977).
- Language is overly casual (Rowland et al., 2011; Sleep & Eskelson, 2012).
- Language forms a maze – an incoherent tangle of words (Smith, 1977).

There were slightly more episodes featuring AWOTS with evidence of verbal imprecision, than without (37 out of 67 episodes) and some episodes featured more than one type of verbal imprecision. The category types and frequency of their presence in the 37 episodes with enacted AWOTS are shown in Table 33.

Table 33. Types of verbal imprecision in AWOT episodes

Type of verbal imprecision	**Episodes (n = 37)**
Language used in incorrect mathematical context	23 (62%)
Language features non-mathematical, informal code	17 (46%)
Language is ambiguous	11 (30%)
Language is overly casual	5 (14%)
Language forms a maze - incoherent tangle of words	2 (5%)

Not every imperfection shown in Table 33 was to do with imprecise articulation of an algebraic way of thinking. Imprecise references to mathematical concepts and procedures were also present in episodes when AWOTS were being explicitly addressed. Those imprecisions were also included in the table because imprecise references of any kind potentially made it difficult for students to make sense of the AWOTS being described.

The two most common types of verbal imprecision noted from all six participants at some point in their lessons when they referred specifically to AWOTS were non-mathematical, informal code and ambiguous language. The preservice teachers used non-mathematical, informal language to describe the mathematical purpose or effect of different procedural steps. Kate, who coincidently also used the overly casual word, "minusing," provided the following explanation to describe the purpose of subtracting 16 from both sides of the equation, $16 + 3y = 10$. She told her class, "If we have plus 16 and minus 16, what do we have? Zero. So by minusing 16, we get rid of him! Make sense?" To make sense of Kate's explanation, students would have had to know that "get rid of him" was an informal code for the undoing part of the doing-undoing way of thinking. According to Falle (2005), this is unlikely to be the case. Falle contends that the statement "to get rid of" is used by secondary students in the context of solving equations to describe "a physical process to remove something which is uneccesary" (Falle, 2005, p. 118) and shows a lack of mathematical understanding. A more precise explanation that spoke of undoing operations and the additive identity would have offered far more mathematical clarity than the catchphrases casually employed by the preservice teachers.

Ambiguous references to the mathematical purpose behind procedures were noted for all preservice teachers. William, for example, attempted to explain to one of his students the reason the expression $\frac{c}{2} - 1 + 1$ was equivalent to $\frac{c}{2}$. Appearing somewhat frustrated with his student's inability to make sense of the written notes he produced, William said, "Minus one plus one is just zero. It means nothing." William's language, "It means nothing", was ambiguous because the meaning of the words, "it" and "nothing," were unclear. Had William been clearer about two inverse operations being performed (the "it" part of his explanation) with no change in the value of the expression (the "just zero" or "nothing" part of his explanation), he might have offered a more precise reference to the "algebraic invariance" way of thinking, where two expressions ($\frac{c}{2}$ and $\frac{c}{2} - 1 + 1$) can take two different forms but be equivalent in value. Once more, the clarity with which the preservice teachers expressed AWOTS was lacking and it seems unlikely that their students would have developed a clear understanding of the lesson content from their explanations.

Preservice teachers often didn't think to enact knowledge of AWOTS. Just over half of the episodes analysed showed no explicit articulation of any algebraic way of thinking.

No preservice teachers’ commentaries referred to intentionally withholding a way of thinking, so it appears that the omissions were missed opportunities. The missed opportunities, the preservice teachers’ surprise at having to explicitly address AWOTS, and the superficiality with which they were often treated during instruction suggests that meaningful treatment of AWOTS seemed to be a rarity on the pedagogical radar of the participants in this study.

The addition of AWOTS in more episodes could have enhanced the mathematical quality of the episodes in general. The complete absence of the “abstracting from computation” way of thinking meant that students focused only on computations involving specific numbers for individual examples and were not encouraged to view multiple examples from a broader perspective (e.g., What is similar about these three examples?). The researcher also repeatedly noted a missed opportunity regarding the algebraic invariance way of thinking which, in the opinion of the researcher, would have significantly improved the quality of the MCK enacted by all preservice teachers at one or more points in their lessons.

All preservice teachers had the opportunity to explicitly teach knowledge of the algebraic invariance way of thinking in multiple episodes but rarely did so. The preservice teachers regularly undertook teaching actions that reflected an implicit awareness that the form of an algebraic term or expression can be manipulated without altering the value of that term or expression. However, they did not share this knowledge with their students and nor did they share the purpose for changing the form of a term or expression as they performed procedures. Sam, for example, wrote the first equation in Figure 13 on the board, when a student asked for help in solving the equation.

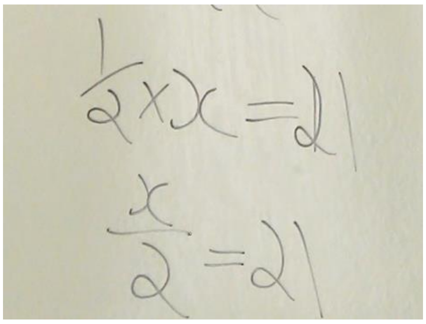


Figure 13. Sam rewrites an equation

After writing up the equation, Sam stood back and thought for a moment. He then added a second equation underneath, saying, “Or we could write this as x over 2 equals 21,” and provided no further explanation for his actions. Sam continued by describing how to solve the new equation but did not explicitly address the circumstances in which an equivalent expression or equation might be desirable or the mathematical benefits of changing the form of the left hand expression to an equivalent expression at that time. From the students’ perspective, his actions would likely have appeared to be random, and without justification, as he simply switched one equation for another.

For the remaining preservice teachers, there were many similar instances when terms or expressions were erased and rewritten, or new versions added without explanation, without the knowledge of an algebraic way of thinking being explicitly shared to accompany and explain the participants’ actions. Missed opportunities such as these were regularly identified in episodes of all preservice teachers’ lessons but in the interviews, they seemed unaware of any limitations regarding ways of thinking. Their lack of awareness suggests particular influences may be contributing to the limitations described.

5.1.4 Possible influences producing limitations in the ways of thinking enacted

The interview comments for episodes featuring limited or notable absences of AWOTS were analysed for possible influencing elements. As preservice teachers did not articulate which influence was impacting their decisions to enact limited forms of AWOTS, a definitive conclusion could not be reached from the data analysed, so some inferences are offered.

The researcher inferred two limitations in the preservice teachers’ own MCK that may have led them to teach limited AWOTS. The first limitation is a lack of MCK and the second is a compressed form of MCK. The interview reflections did not suggest that they held considerably deeper knowledge or more precise mathematical language. Their descriptions of AWOTS in the interviews generally mimicked their own descriptions in the lesson, with the same types of verbal imprecision and shallow mathematical depth. The nature of the stimulated recall interview technique used in this study prevented the researcher from asking more specific questions to gauge the preservice teachers’ knowledge of any ways of thinking in a more detailed manner. However, the notable absence of more precise or thorough versions of knowledge across the interview reflections seems to indicate that preservice teachers did not draw on a deeper knowledge of AWOTS. It seems likely that the preservice teachers' MCK might have been incomplete and they may have lacked certain language or mathematical connections that would have allowed them to address the ways of thinking more successfully.

A second potential limitation is the compressed form in which preservice teachers hold their knowledge of AWOTS. In the interviews, the preservice teachers gave little indication that they were aware of particular ways of thinking, yet their classroom actions and talk about their actions regularly implied their reliance upon these ways of thinking. It would not be reasonable to suggest that the preservice teachers had no knowledge of certain AWOTS because they were observed in a tacit form as they performed many procedures. What seems more likely is that preservice teachers do not hold their knowledge of AWOTS in an accessible form so they are unaware of what ways of thinking they are drawing upon as they perform certain algebraic procedures.

It is possible that another influence behind limited AWOTS is the preservice teachers' judgements about students. The preservice teachers' judgements may not be developed enough for them to recognise the value of sharing a more precise and connected knowledge of AWOTS with their students. For preservice teachers to consider the value of attending to certain AWOTS, however, they would need to be aware of those ways of thinking, the associated language, and how they connect to algebraic manipulations. The preservice teachers' thoughts and actions in this study did not indicate this awareness. The underdeveloped nature of the preservice teachers' MCK may therefore inhibit the preservice teachers' ability to know the mathematical content to which their students should be exposed. While an absolute conclusion cannot be drawn from the data, it seems likely that deficiencies in the preservice teachers' MCK and perhaps their student judgements too are an influence on the MCK they deliver in multiple episodes that is devoid of meaningful attention to AWOTS.

5.2 Preservice teachers enact conceptual knowledge

We don't just move things! I've told them a thousand times and still, "Well, we move the four." No, we do not! We use opposite operations to manipulate the equation...

(Ben, talking about solving equations)

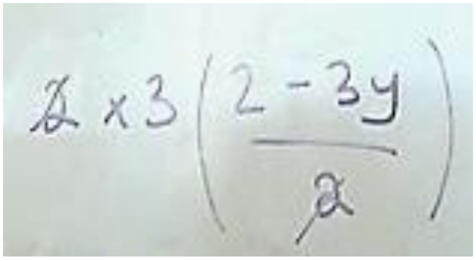
5.2.1 The conceptual knowledge enacted

The preservice teachers' knowledge of mathematical concepts manifested only occasionally in their classroom actions and was noted in 48 of the 137 episodes (35% of all episodes). Conceptual knowledge was rarely enacted in isolation (7 episodes) and tended to be presented alongside either procedural knowledge (6 episodes), AWOTS (5 episodes), or both other knowledge types (30 episodes). Knowledge of algebraic objects and arithmetic operations comprised the conceptual knowledge that preservice teachers explicitly referenced during their lower secondary algebra lessons. Table 34 provides a summary of the type of conceptual knowledge that the participants taught. The total percentage of episodes exceeds 100% because the preservice teachers sometimes chose to enact two different types of conceptual knowledge in the one episode.

In the same way that AWOTS manifested as brief supporting statements within explanations of procedures, conceptual knowledge was also referred to quite briefly by the preservice teachers. No preservice teacher spent significant time during a lesson addressing a mathematical concept. Instead, momentary references to mathematical concepts were interspersed amongst longer procedural explanations, resulting in few instances where strong, explicit connections were made between the procedures and the related concepts.

Specialised conceptual knowledge was evidenced in only 18 of the 48 episodes featuring conceptual knowledge. Specialised conceptual knowledge was always taught together with procedural knowledge (100% of the 18 episodes) and usually with AWOTS (89% of the 18 episodes). The preservice teachers taught a specialised knowledge of mathematical concepts when they made explicit connections between a concept and a related procedure or rule, which are considered pedagogically valuable actions (Kilpatrick et al., 2001; Ma, 1999; Skemp, 1979).

Table 34. Aspects of conceptual knowledge evidenced in episodes

Conceptual knowledge	**Episodes (n = 48)**	**Episode excerpts**
Knows a feature of a term that is a product	11 (23%)	"So when there's no sign there, you know it's a multiply, so it's just two x." (Kate, talking about the term, $2x$)
Knows a feature of an algebraic equation	10 (21%)	"It means it's like the same on both sides. So we're saying that this half is the same as this half." (William talks about the equals symbol in $a + 6 = 13$)
Knows concept of the inverse operation	10 (21%)	"These two here will cancel, yes? I've got a times by two and a divided by two." (Grace simplifies the expression below) 
Knows a feature of a term that involves division	7 (15%)	"We've got x on six, correct? Which is x divided by six, the same thing." (Ben)
Knows a feature of an algebraic term	6 (13%)	"The numerical part of a term that contains a variable is called the *coefficient*." (Thomas' written board work)
Knows conventions regarding order of operations	6 (13%)	"And then when we go backwards… BOMDAS, yeah? We go backwards in our order of operations." (Grace talks about solving $-30 + 17y = 21$)
Knows a feature of an algebraic expression	2 (4%)	"When a plus or minus sign separates an algebraic expression into parts, each part is a term." (Thomas' written board work)
Knows two representations of addition operation (word and symbol)	2 (4%)	"Pretend you're Emily and your age is e. In ten years' time, your age will be what it is now plus ten years. So it's going to be e plus ten." (Sam, who writes down "$e + 10$")
Knows two representations of subtraction operation (word and symbol)	2 (4%)	"We know the difference is x minus y." (Kate, referring to the expression, "$x - y$")

Across the 18 episodes where explicit connections between concepts and other MCK were made, three concepts were given attention by the preservice teachers: inverse operations (10 episodes), equivalence (7 episodes), and conventions governing the order of operations (6 episodes). Each concept and its connection with other MCK was communicated with care in some cases and more casually in others. The examples of teaching actions below show how the preservice teachers were beginning to make deeper connections involving conceptual knowledge, albeit with limitations which are discussed in section 5.2.3.

All participants enacted conceptual knowledge of inverse operations during their lessons. William devoted a substantial amount of time in his algebra lesson to developing the connection between inverse operations and their role in undoing operations performed on a pronumeral, discussed in section 5.1.1. He told his students, “I’m thinking of a number in my head, and when I add six to the number, the answer is 13” (Appendix O).William then asked his students to work out the value of the number. When students quickly responded with the correct value, William then attempted to focus on the solving process they had intuitively used by presenting questions, verbal prompts, and responses that repeatedly drew their attention to “doing the opposite”. Although his explanations were neither comprehensive nor eloquent, William was drawing upon a form of specialised knowledge of the solving process in the episode by making an explicit connection between the concept, inverse operations, the doing-undoing way of thinking, and a procedure to solve a linear equation.

Equivalence was another concept that was highlighted by the preservice teachers when they enacted specialised conceptual knowledge. Of all the preservice teachers, Ben explicitly addressed the concept of equivalence and solving equations the most often with his students and he did so with visual and verbal cues. Ben loved balance beams. He drew them so often under equations in his board work (an example was provided in section 4.2.3, Figure 5) that his students knew to suggest them if they couldn’t come up with any other ideas on how to begin solving. After one confused student suggested “balance beams?” as a first step in solving the equation, $3x + 4 = 2x - 1$, Ben added them under the equation, joking with his class, “Yup. Good start. It’s always a fail safe. If you don’t know what to do, say balance beams and you get a tick.” Ben’s emphasis of equivalence did not end with the visual cue provided by the balance beams. He supplemented the visual cue by verbally reinforcing the concept of equivalence, using statements such as, "So we need our balance beam, because we need equivalence, don't we?" The following excerpt from one episode illustrates the connections Ben emphasised when reviewing the solving process with his students as they made suggestions as to how to go about solving the equation, $4x^2 = 16$.

Student:	Divide.
Ben:	Yup. So do we divide one side by four, or what?
Student:	Divide both sides.
Ben:	Yup. Why?
Student:	Keep it even.
Ben:	Keep it even. Keep our equivalence.

More reminders were interspersed amongst the procedural steps Ben presented, so the connection between performing the same operation to expressions on each side of an equation and conservation of equivalence was explicitly communicated a number of times. Although Ben did not make connections to an algebraic invariance way of thinking or conceptual notions such as equivalent equations, he nevertheless was able to draw upon a pedagogically valuable connection involving conceptual knowledge in his teaching actions.

The concept of order of operations was drawn upon by three of the preservice teachers as they explicated particular solving methods to their students. Connected knowledge about order of operations and procedural steps was most carefully articulated by Sam, but with quite strict limitations on its application in a solving procedure, which is discussed in more detail in section 5.2.3. When Sam's supervising teacher noticed that his students were having trouble identifying the "correct" operation to undo when solving two-step linear equations, Sam took his supervising teacher's advice and taught a backtracking method in his second observed lesson. He presented the equation $2x + 7 = 17$ to his students and took care to connect the steps of the backtracking method (procedural knowledge) to the conventions regulating the order of operations (conceptual knowledge), and thus, enacted specialised content knowledge. See Appendix P for details of his approach. To better understand why Sam and the other preservice teachers taught conceptual knowledge, the decision making thoughts behind their actions were investigated.

5.2.2 Influences associated with conceptual knowledge

This section describes the influences on preservice teachers' decisions to enact conceptual knowledge in an episode. The analysis of influencing elements that resulted in episodes featuring conceptual knowledge (n = 48) and specialised conceptual knowledge (n = 18) revealed the following three sets of influencing elements:

- Episode goals that emphasise content beyond mastery of procedures or that address student confusion;
- Small instructional settings and interactive decisions;
- Lesson phases where content is consolidated and interactive decisions.

5.2.2.1 Episode goals leading to conceptual knowledge

Enacted conceptual knowledge was associated with four types of content and student focused episode goals. The preservice teachers' episode goals, shown in Table 35, reflect the attention they paid to the lesson content and to their students' points of confusion when they decided to deliver conceptual knowledge.

Table 35 shows that when preservice teachers formed the content focused goal, "Teach students appropriate use of mathematical language," they tended to enact common content knowledge of concepts. Conceptual knowledge was discerned in 81% of episodes where preservice teachers' acted with this episode goal in mind. However, only one of this episode goal type led to specialised conceptual knowledge. This shows that the conceptual knowledge presented with this episode goal in mind was common content knowledge. The preservice teachers' aim in these episodes was for their students to write algebraic terms in appropriate ways, such as representing division using a vinculum or omitting the multiplication symbol. For example, Sam reflected for one episode, "I'm trying to get them to write it algebraically, so $2k$. Just because that's convention, how we write things."

Table 35. Type and relative frequency of episode goals for conceptual knowledge episodes

Major category of episode goals	Type of episode goal	Total goals for type (n = 174)	Goals for conceptual knowledge		Goals for *specialised* conceptual knowledge	
			No. (n = 66)	Percentage of total goals by type	No. (n = 24)	Percentage of goals for conceptual knowledge by type
Content focused goals	Develop students' knowledge of procedures	60	8	13%	3	38%
	Teach students appropriate use of mathematical language	16	13	81%	1	8%
	Connect procedure with a concept	12	8	67%	7	88%
	Associate procedure with certain types of solutions	10	3	30%	0	0%
	Connect procedure with mathematical purpose	9	5	56%	4	80%
Student focused goals	Address student confusion	32	18	56%	6	33%
	Value and/or encourage student contribution	12	3	25%	2	67%
	Gauge student knowledge	12	3	25%	0	0%
	Avoid student confusion	11	5	45%	1	20%

The content focused episode goals which tended to lead preservice teachers to present more specialised forms of conceptual knowledge were those to do with mathematical connections. In 67% of episodes underpinned by an episode goal to connect a concept to a procedure and in 56% of episodes where the preservice teachers wanted to connect a procedure with its mathematical purpose, conceptual knowledge was enacted (see Table 35). The majority of goal types concerning connections with concepts (88%) and mathematical purpose (80%) led to episodes with more connected and therefore specialised forms of conceptual knowledge enacted. For example, Ben reflected on an episode featuring specialised content knowledge, commenting, "If they can understand equivalence…I'll be happy. I think their knowledge about algebra will be stronger." This shows a positive association between a preservice teachers' goal to make a mathematical connection and teaching actions that include explicit references to connections involving conceptual knowledge.

Preservice teachers decided to teach conceptual knowledge when they assisted confused students. Over half (56%) of all episode goals to "address student confusion" resulted in preservice teachers sharing an aspect of conceptual knowledge with their students (see Table 35). For example, a student of William insisted that the solution to the equation, $x + 3 = 10$, was 13. The student's response prompted William to sit with him for a few minutes and talk about the effect of adding and subtracting the same value to and from a pronumeral (i.e., inverse operations). In William's case and for all other preservice teachers, interactions with confused students tended to lead them to call on conceptual knowledge in their explanations. The instructional setting where interactions such as these took place was an additional influence on enacted conceptual knowledge.

5.2.2.2 Smaller instructional settings and interactive decisions

Instructional settings involving a small number of students were associated with episodes featuring conceptual knowledge. The type of instructional setting for episodes featuring any conceptual knowledge and for specialised conceptual knowledge are presented in Table 36.

Table 36. Conceptual knowledge episodes by instructional setting

Instructional setting	Total episodes for type	Conceptual knowledge episodes		*Specialised* conceptual knowledge episodes	
		No.	Percentage of total episodes for type	No.	Percentage of conceptual knowledge episodes
Whole class	116	39	34%	14	36%
Small group or individual	13	6	46%	3	50%
One student, conducted in front of class	8	3	38%	1	33%
Total	137	48	35%	18	38%

Table 36 shows that almost half of all episodes that took place in a small instructional setting, that is, with either one student or a small group of students, resulted in the participants teaching conceptual knowledge. Although the relative frequency highlighted in the table is slightly below 50% for all episodes (n = 137), the result is still significant, given that only 35% of all episodes (n = 48) featured conceptual knowledge. Further examination of the six episodes highlighted in the table revealed that each one was the result of an interactive decision and was underpinned by the episode goal to address student confusion. These results show that when preservice teachers are responding to student contributions in a small instructional setting, they tend to identify certain points of student confusion and then address them by enacting conceptual knowledge.

5.2.2.3 Lesson phases to consolidate content and interactive decisions

The preservice teachers tended to deliver conceptual knowledge more often in some types of lesson phases than in others. The proportion of episodes featuring conceptual knowledge are shown in Table 37 according to the lesson phases in which they were located and whether they were the result of a live classroom event.

Table 37. Conceptual knowledge episodes by lesson phase and classroom event

Lesson phase	Total episodes for type	Conceptual knowledge episodes		Conceptual knowledge episodes by classroom event	
		No.	Percentage of total episodes for type	No event	Classroom event
Introduce content	17	7	41%	5	2
Consolidate content	43	18	42%	8	10
Develop content	20	4	20%	3	1
Review content	57	19*	33%	14	5
Total	137	48	35%	30	18

**Ben was responsible for 14 of the 19 episodes located within a review phase*

The high proportion of "review content" episodes in Table 37 are the result of only one preservice teacher, Ben, who taught 14 of the 19 episodes in this phase. It is not reasonable to say, therefore, that the participants tended to enact conceptual knowledge when they reviewed mathematical content because most preservice teachers did not. The majority of participants did, however, present conceptual knowledge in lesson phases where they aimed to consolidate their students' knowledge of lesson content. In that lesson phase type, they taught conceptual knowledge more often as the result of a spontaneous MCK related decision, usually prompted by a student contribution (nine of the ten classroom events prompting consolidate content episodes), than a pre-planned one. Student contributions appeared to reveal the need for preservice teachers to explicitly address conceptual knowledge to add meaning to the procedures students had been exposed to but did not fully understand. It seems that student prompted situations where preservice teachers aim to consolidate their students' understanding of mathematical content by delivering whatever MCK the situation calls for, potentially provide fertile ground for preservice teachers to decide to teach conceptual knowledge.

5.2.3 Limitations in the conceptual knowledge enacted

Conceptual knowledge that is presented alongside procedural knowledge encourages students to develop a relational understanding of mathematics (Skemp, 1979). All preservice teachers enacted conceptual knowledge during their lesson in an effort to develop their students' relational understanding of algebraic procedures by explaining why certain procedural steps could be performed. The preservice teachers' attempts were hampered by three limitations, described in detail in this section. The limitations comprised verbally imprecise explanations, a failure to make explicit the contextual restrictions in the way mathematical concepts were dealt with in lessons, and notable absences of pertinent conceptual notions.

Just as the preservice teachers' mathematical language was problematic with AWOTS, so, too, was it a concern in the episodes featuring conceptual knowledge. In the 48 episodes in which the preservice teachers communicated their knowledge of mathematical concepts, verbal imprecision was noted in 20 of them. Not all imprecise language referred specifically to the concepts discussed. However, the presence of poorly expressed mathematical statements in the episodes with conceptual knowledge would have made it difficult for students to fully understand the conceptual points the participants were trying to make. Analysis of the preservice teachers' specific references to mathematical concepts during the episodes revealed that a limitation common to all six preservice teachers was the use of ambiguous and contextually imprecise language to communicate certain concepts.

Ambiguous language featured in the classroom talk of all preservice teachers when they found ways to discuss mathematical concepts without ever referring to them directly. Extremely vague references to concepts such as "this bit", "get that to zero", or "thing" replaced more specific references to concepts in the preservice teachers' language. The ambiguous language resulted in explanations such as, "You had to do two separate things to get c by itself" (William) or "The unknown, k. That's what we're doing things to" (Sam). Even when Thomas presented a glossary to help his students become more familiar with particular language used to describe algebraic objects (e.g., term, coefficient), he made no reference to language concerning the pronumeral in his board work. As he spoke to the class, Thomas brushed over the language students would need to talk about the pronumeral itself, saying, "At the moment, the x and the y, we don't know what they are, they're just variables, they're unknowns. So don't worry about that too much." Thomas made no further remarks about the mathematical contexts in which a pronumeral would be considered a variable or an unknown, so his explanation was contextually imprecise. The absence of more precise descriptions of mathematical concepts meant that mathematical meaning was potentially difficult to come by for the preservice teachers' students and subsequently limited the conceptual knowledge that manifested in the preservice teachers' verbal descriptions.

Five of the preservice teachers were observed presenting conceptual knowledge that was limited in its application beyond the lesson content (Kate, Ben, Grace, Sam, and Thomas). The notion of "trimming" (McCrory et al., 2012, p. 604), described in more detail in chapter 2, refers to reducing content in algebra lessons to suit the cognitive level of school students. However, McCory et al. (2012) warn that in doing so, it is possible to trim algebraic content too much and lose important details. Delivering overly reduced content within one lesson that does not accurately reflect a broader view of mathematics beyond the lesson context is known as sacrificing the mathematical integrity of the content presented (Lim, 2008; Ma, 1999; Schifter, 2001; Wu, 2006). Although the concepts shared by the preservice teachers made mathematical sense within the confines of the content presented in the lesson, important details were omitted, distorting how students might make sense of the lesson content if applied to more advanced mathematical contexts.

The restricted treatment of certain mathematical concepts reflected the preservice teachers' willingness to sacrifice the mathematical integrity of the content they delivered. For example, when Ben asked his class, "So what's the square root of four?" as he solved the equation, $4x^2 = 16$, he accepted a student's response of two as the only solution to the equation, although he commented in the interview that he was aware that another solution existed. In this instance, Ben chose to deliver a limited version of his own MCK, but in doing so, he potentially limited his students' perceptions about mathematical roots and the number of solutions that are possible when solving quadratic equations.

Sam and Grace also limited the conceptual knowledge they chose to enact by restricting the way in which a mathematical concept can be used to support a solving procedure. Sam repeatedly referred to the mnemonics 'BOMDAS' and 'SADMOB' as mandatory principles which must be used to solve equations. Similarly, Grace spoke of the order of operations when she talked to her class about solving the equation, $-30 + 17y = 21$, saying "And then when we go backwards… BOMDAS, yeah? We go backwards in our order of operations. So instead of doing times and divide by first, we do plus and minus first when we're solving equations." The participants did not refer to the order of operations as a concept that may be used to identify an efficient solution path in particular cases, but instead, as a mandatory concept that must be applied in all solving procedures.

For example, in an equation such as $-30 + 17.5y = 21$, it would be reasonable to multiply both expressions on either side of the equation by two as a first step and for equations such as $-30 + 17y = 21y^2$, adding 30 to each side of the equation would not be a sensible choice. Whilst the restricted treatment of the order of operations concept could be used to solve the simple linear equations presented in the lessons, the overly trimmed conceptual knowledge that Sam and Grace presented may have distorted their students' view of the necessity and suitability of the concept's application in other contexts. A connection was certainly articulated by Sam and Grace of a concept that underpinned a procedure but a less compulsory handling of the connection would have encouraged a more flexible approach to applying the order of operations conventions when solving equations. Interestingly, both Sam (but not Grace) in these examples and Ben (but not Thomas) in the previous examples were aware that they were restricting the MCK they delivered but did not consider this a problem. Their reasons for doing so are discussed in section 5.2.4.

Missed opportunities to teach conceptual knowledge that would have enhanced the episode were identified for every participant in every lesson. Rules and steps without reason, such as Grace's reminder, "What I do to this side [of an equation], I must also do to this [other] side," were commonplace in many episodes. With the exception of Ben, who missed a number of conceptual opportunities himself, the absence of a conceptual explanation in one episode was rarely balanced with one in a neighbouring episode. Kate, in particular, encouraged her students to apply a number of conceptually free shortcuts. At one point in her first lesson, after her students had spent some time solving simultaneous equations using the substitution method, Kate introduced a graphing software program and suggested her students enter in the two equations and find the intersection point. She told her students, "So when you're doing your equations in your book, if you're not sure you have the right answers, you can just plug in any two equations and find your intersection point. It's magic!" Kate bypassed the opportunity to make a very strong connection between a solution in graphical and symbolic forms, and instead, promoted the software program as a superficial way to check the students' symbolic manipulations. Conceptual knowledge in this episode and in many others would have been a valuable addition but was absent.

Notably absent from many episodes was explicit attention to the process/object duality espoused by scholars such as Sfard (1991) and Gray and Tall (1994). Sfard (1991) argues that abstract mathematical ideas can be thought of "in two fundamentally different ways: *structurally*, as objects, and *operationally*, as processes" (p. 1). Two mathematical notions, the minus symbol and the equals symbol, and their treatment within the lower secondary algebra lessons offered the preservice teachers the opportunity to explicitly address this duality but none chose to do so, to the potential detriment of their students' understanding.

The dual meaning of the minus symbol was one that was never explicitly addressed by any of the preservice teachers. Within a number of episodes, the preservice teachers moved swiftly and easily between the two possible functions of the symbol, first as a binary function, where the symbol represented subtraction (symbol as a process), and second, as a unary function that "makes a number negative" (Vlassis, 2004, p. 472). It was not uncommon for them to use both meanings even within one sentence, as Thomas did when expanding the brackets in the expression, $5\,(x - 12)$, to produce $5x - 60$. He told his students, "So we've done five times x, five times *negative* 12, which is $5x$ *minus* 60." In this instance, Thomas appears to have interpreted the minus sign first as addition of a negative number, and moments later, as a binary operator. He failed to explain this distinction to his class, just as he failed to mention that he had reinterpreted the expression within the brackets as a sum, to use the distributive law of multiplication over addition, rather than over subtraction, which was also possible. All preservice teachers had multiple opportunities to make this distinction but they either could not or chose not to do so. Their lack of comments regarding the dual nature of the minus symbol in the interview, with few exceptions, meant that it was not possible to determine why the participants didn't pay more attention to this concept during instruction.

The second process/object interpretation that the preservice teachers implicitly drew on but failed to explicitly address was related to the equals symbol and the structure of an equation. Knuth et al. (2006) note the dual function of the equals symbol first, from an operational point of view, where a process (operating on a pronumeral) has produced a particular result (equation as a process), and second, from a relational view, where an equals sign is treated as a symbol of mathematical equivalence (equation as an object). Five preservice teachers (William, Sam, Thomas, Kate, and Ben) did not explicitly address the dual nature of the equals sign when discussing how to create equations from different worded scenarios. Appendix Q presents examples of worded problems presented in the preservice teachers' lessons, according to the interpretation (operational or relational) needed to represent the problems with equations. The opportunity to address the dual meaning of the equals sign was evident in all preservice teachers' lessons but they did not share this knowledge with their students.

5.2.4 Possible influences producing limitations in the conceptual knowledge enacted

The interview reflections were examined for particular influences that appeared to be contributing to the limited conceptual knowledge presented by the participants. The interview reflections revealed that the preservice teachers were not usually aware that they were teaching limited versions of conceptual knowledge and made no reference to any conceptual limitations in their teaching. The episode reflections suggested limitations in their own MCK impacted their decisions to share limited versions of conceptual knowledge. Three preservice teachers did reflect at one point in their interviews that limiting the conceptual knowledge they shared with their students was a conscious decision. For those preservice teachers who knowingly limited what they knew about mathematical concepts during instruction, the judgements they held about their students appeared be a significant influence on their decisions.

The limited expressions of conceptual knowledge that manifested in the participants' actions were regularly repeated in the interviews. The preservice teachers, at times, admitted to a lack of mathematical language needed to provide appropriate explanations of concepts to their students. William, Sam, and Ben all admitted to struggling to find the right words to offer their students when referring to different mathematical concepts. William, for example, reflected on his difficulties when told by his mentoring teacher to avoid the use of a mnemonic to describe the order of operations concept. His reflection revealed a lack of language to explain the order in which operations can be undone, without making reference to BODMAS. He said,

> I knew in my head that we had to do these addition and subtraction before the multiply and divide but I didn't know how to explain that, I guess. And I had just spoken with my teacher and she said that she hated BODMAS and all those little

sayings. She hated BODMAS so I was thinking, "How do I explain it without using what she has just told me not to use?"

In William's lesson, he avoided using the term, BODMAS, but also chose not to make any reference whatsoever to the order in which he was undoing operations. He simply told his students that he was showing them the "easiest way" to solve equations. William's reflection suggests that knowing how to apply a mathematical concept within a procedure and possessing a comprehensive knowledge of that concept, including the language to describe that concept, are different facets of conceptual knowledge. A lack of the latter knowledge type appeared to be evident in this case.

Not all preservice teachers openly admitted to lacking the mathematical language needed for teaching. However, it does seem likely that the preservice teachers do not appear to hold complete and unpacked notions of certain mathematical concepts, including appropriate terminology, related to lower secondary algebra content. For example, in their interviews, all preservice teachers either avoided any discussion of the meaning of the equals symbol or failed to provide comprehensive or clear descriptions. Both Kate and Sam, for example, did not mention throughout their lessons that certain worded problems and equations were structured differently to others, even though their students appeared to be finding scenarios requiring a relational understanding of the equals symbol more difficult to translate into symbols. In Kate's interview, she described those questions as more challenging but was unable to articulate the concept underpinning the difficulties. Sam had limited success describing a relational view of the equals sign needed for certain questions in his interview, as he noted of question 3 in Appendix Q, "This one they had to interpret these things were actually going to be equal," and "You kind of have to develop both sides at the same time in your head." The lack of clarity regarding worded problems and equal symbol interpretations that both preservice teachers provided in their interviews was echoed by the other four preservice teachers, none of whom were able to identify both views (operational and relational) of the equals sign successfully, suggesting an underdeveloped knowledge of this concept. Their descriptions of the minus sign, however, showed an emerging awareness of the process/object duality for a symbol.

The process/object duality of the minus symbol was referred to a number of times in related episode reflections. Often prompted by memories of confused student contributions, the preservice teachers discussed the minus symbol with differing degrees of awareness in their interviews when they explained the mathematical troubles that their students were encountering when operating with integers. Thomas, for example, referred to the minus symbol as "negative" and "minus" within one mathematical statement in the episode where he was manipulating the expression, $5\ (x - 12)$, reported in section 5.2.3. In his interview, he again referred to both views of the minus symbol, reflecting, "Kids go, 'But it's not negative 12, it's just 12. And it's take away?' And I'm sort of, like, 'Well, you're right…and that throws them off so much." Thomas did not demonstrate any knowledge of the additive inverse or an awareness that he was reinterpreting "subtract 12" as "add negative 12" either in the lesson or the interview. Sam was not even able to articulate a muddled response similar to Thomas when his students were having trouble adding and subtracting integers and could only remark with frustration in the interview, "You just add them!". These preservice teachers, who appeared so relaxed in class reinterpreting any positive number to be subtracted as equivalent to adding its opposite, were unable to separate the two meanings of the symbol in their interviews. Their difficulties suggest that their MCK in this instance was too compressed for them to access and present with clarity to their students or the researcher.

Relatively more unpacked MCK was evidenced in two other participants' reflections, but they both found it difficult and frustrating to articulate the specific process/object duality of the minus symbol. William and Ben both referred in the interview to the dual nature of the minus sign, but their references were not fully developed. William, for example, was aware that his students were having difficulties operating with negative numbers and pronumerals, so he intentionally provided the review question shown in Figure 14 to his class (see Appendix R for details of the episode).

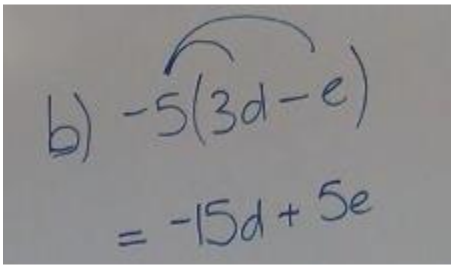


Figure 14. William's review question

In the episode, William told his students, "We're multiplying negative five times negative e," failing to mention that he was now treating the subtraction sign as addition of the term,

“-e.” The structure of the expression inside the brackets was not addressed to highlight either the reinterpretation of the subtraction sign as addition of the additive inverse of the pronumeral or to highlight that the distributive law of multiplication over addition rather than subtraction could be employed for this example. William reflected in his interview that the continuing problem he thought his students were encountering was caused by their interpretation of the minus sign as a subtraction symbol and not as “plus a negative.” It seemed that William had a sense of the conceptual issue that was likely to be causing his students’ confusion, so his MCK appeared to be a little less compressed than that of other preservice teachers. Both he and Ben, however, acknowledged that they were having trouble understanding exactly what their students’ problem was, so their MCK was not yet in a form that allowed them to pinpoint the concept specifically enough to adequately address it in their lessons.

Three preservice teachers relied upon their judgements about students to inform their decisions to knowingly enact limited forms of conceptual knowledge during instruction. Ben, Sam, and William acknowledged in their interviews that they intentionally restricted the conceptual knowledge they shared in the interests of their students. Ben admitted that he withheld one of the solutions to the equation, $4x^2 = 16$, reflecting, “I thought about trying to expand it with that but with square roots I thought I’d get pretty … backs to the wall and I wanted to build some confidence before their exam.” Sam spoke in his interview about his insistence that students exclusively use conventions involving the order of operations (SADMOB, as suggested by his mentoring teacher) when solving equations. He reflected that when he solved equations himself, he didn’t think of BODMAS or SADMOB, saying, “I don’t think of it at all. I think for the kids it’s the best way.” William was the only preservice teacher who spoke with significantly more mathematical precision in his interview comments than in his lesson. He talked about having to “dumb down” the way he usually spoke about algebra, justifying his decision as follows:

> I guess, my thought process was that for some kids, this is going to be hard enough, understanding algebra. I don’t want to throw in more complicated words that are going to be something else they have to be working out, “Well, what does that word mean?”, and thinking about that, distracting them from the topic of just algebra. That’s why I was saying, “How do we get rid of this?” and “What’s the

> opposite?", and I was keeping my language very basic and in a way that they speak… I was trying to keep it very basic, so the only thing they had to be thinking about was the algebra.

For William, Ben, and Sam, their judgements about the content to which students need exposure and about how students learn mathematics contributed to their limiting of the conceptual knowledge they chose to share explicitly with their students. It is also possible that the preservice teachers lacked the horizon knowledge needed to trim algebraic content adequately. The MCK and judgements about students called upon by preservice teachers as they make decisions regarding conceptual knowledge seem to be contributing to the limited versions of conceptual knowledge that manifest in their teaching actions.

5.3 Preservice teachers enact procedural knowledge

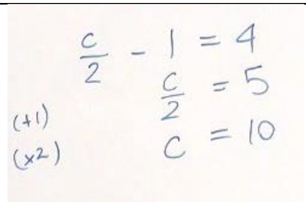
That's why I had these questions lined up, 'cause I knew they were going to have to expand the brackets and collect the like terms…so it would cover those. (Thomas)

5.3.1 The procedural knowledge enacted

In every lesson, by every preservice teacher, and in the vast majority of episodes, procedural knowledge was enacted. Of the 137 episodes, 125 (91%) revealed evidence of the preservice teachers presenting procedural knowledge. The overall effect of such a high percentage of episodes featuring procedural knowledge was that verbal and written referrals to procedures dominated the talk and board work of the preservice teachers. The procedural knowledge discerned from the episodes was categorised to identify what, in particular, was emphasised about algebraic procedures. Table 38 shows the classifications that arose from the analysis. The total percentage exceeds 100% because the preservice teachers regularly presented more than one facet of their procedural knowledge within an episode.

"Follow the steps" was the preservice teachers' mantra at regular intervals in every algebra lesson. Table 38 shows that knowledge of the steps and rules of mathematical procedures dominated the procedural knowledge that manifested in the preservice teachers' actions.

Table 38. Aspects of procedural knowledge evidenced in episodes

Procedural knowledge	Episodes (n = 125)	Episode excerpts
Knows how to perform a written procedure	64 (47%)	 (William's board work)
Knows a verbal explanation for a procedure	57 (42%)	"So we have x divided by two. To get rid of two on this [left hand] side, what do we do? Multiply by two on this [right hand] side. So x equals 21 times two. See what I did there?" (Sam talks about solving the equation, $\frac{x}{2} = 21$)
Knows a mathematical rule that must be followed in a procedure	44 (32%)	"What I do to this side, I must also do to this side." (Grace, in the context of solving equations, gesturing to each side of an equation)
Can produce an equation or expression to which a certain procedure can be performed	31 (23%)	Expand the brackets: $2(x+1)$, $5(4-x)$ | Find x: $2x + 8 = 16$, $\frac{x}{5} - 9 = 6$ (Thomas' board work)

There was evidence that the preservice teachers also delivered specialised forms of knowledge about mathematical procedures in 35 episodes (28% of episodes featuring procedural knowledge). Ma (1999) contends that specialised mathematical knowledge for teaching requires breadth and depth and the preservice teachers enacted broad and deep procedural knowledge in three ways. Firstly, they demonstrated a breadth of knowledge when interpreting students' alternate solution paths and working flexibly with multiple solution paths (7 episodes). Secondly, they presented a specialised, deep knowledge of smaller sub-procedures that were embedded within larger procedures (10 episodes). Thirdly, the preservice teachers explicitly demonstrated their awareness of example features that created more or less mathematically demanding procedures than others (21 episodes). Awareness of the mathematical complexity in an example is a key feature of specialised content knowledge according to Ball et al. (2008) and reflects the care that preservice teachers regularly took when they mediated procedural knowledge with their classes.

The preservice teachers enacted broader (Ma, 1999) or wider (Rowland et al., 2009) MCK needed for teaching, evidenced in their flexible approach to algebraic solution paths (Yakes & Star, 2011). At one or more times in their lessons, five of the six preservice teachers were given a suggested procedural step by their students which did not align with the solution path they already had in mind. There was an ease with which preservice teachers were able to follow the procedural "train of thoughts" of their students, reflecting a broad knowledge of algebraic manipulations that they were able to draw upon when deciding how to respond. In their interviews, the five preservice teachers were comfortable describing their own procedural steps and those of their students, providing further evidence that they possessed broad procedural knowledge that extended beyond what a non-teacher would need to complete a procedure only in their preferred way. The type of MCK related responses they gave to their students' suggestions, however, were markedly different.

The opportunity to communicate a specialised knowledge of procedures by following an alternative solution path was disregarded by some preservice teachers and reservedly welcomed by others in the classroom. The alternate solution paths that preservice teachers briefly acknowledged but intentionally avoided discussing in detail with their students are explored in section 5.3.3 as they represent intentional omissions of MCK. Kate and Ben did choose to communicate their knowledge of alternative procedural steps with their class but not with a great deal of enthusiasm because they were unsure about the wisdom of deviating from what they considered the best solution path for their students. The influences impacting their decision to accommodate their student requests and take alternative solution paths, which are reported in section 5.3.2, provide an insight into the tensions experienced by the preservice teachers when they possessed certain MCK but were unsure about sharing what they knew with their students.

Careful attention was paid by all preservice teachers to the development of students' knowledge of sub-procedures that together formed larger procedures. Essential features of algebraic expressions and equations were also highlighted by all participants in one or more episodes. Sam, for example, presented the linear equation, $8x - 4 = 6x + 2$, for his students to solve, recognising that x was located on both sides of the equation. The backtracking procedure he had just presented to his students required x to be only on one side of the equation, so Sam introduced a sub-procedure to manipulate the equation into a form where the backtracking method could then be employed. His explanation to his students was as follows:

> We've got x's on both sides of the equation. That makes it a bit tricky to work out what x is? So the easiest thing to do is to get x on one side of the equation because, over here (walks over and gestures to backtracking box method example), we've got x on one side of the equation. So we'll get 'em all to one side.

Sam then modelled the sub-procedure of transposing terms from one side of an equation to the other side not as a solitary skill but as part of an overall procedure. The emphasis by the preservice teachers on sub-procedures and algebraic features in their episodes reflects a developing specialisation of the procedural knowledge they possess and enact for teaching.

5.3.2 Influences associated with procedural knowledge

The ubiquitous presence of procedural knowledge called for a variation in the way episodes featuring procedural knowledge were analysed. Of the 125 episodes where procedural knowledge was discerned, 57 featured only procedural knowledge and no conceptual knowledge or AWOTS. Those 57 episodes and the accompanying reflections were analysed by the researcher to identify influences that led preservice teaches to specifically enact procedural knowledge. A second subset of the 125 episodes featuring procedural knowledge was created to explore the thoughts behind preservice teachers' decisions to enact specialised forms of procedural knowledge. The specialised procedural knowledge subset of 35 episodes comprised 11 of the episodes featuring only procedural knowledge and 24 episodes with either conceptual knowledge and/or AWOTS. The remaining 44 episodes with procedural knowledge do not form part of the analyses reported in this section as influences associated with these episodes were reported in sections 5.1 and 5.2. In this chapter section, influences behind (a) procedural knowledge enacted in isolation and (b) specialised procedural knowledge presented with or without other knowledge types are reported.

The analysis of the preservice teachers' decisions to teach only procedural knowledge revealed two very different influences from those leading to enacted ways of thinking or conceptual knowledge. The following two sets of influencing elements resulted in procedural knowledge in isolation:

- Lesson and episode goals that emphasise procedures and their solutions or that focus on students' mathematical understandings;
- Preactive decisions, impacted by judgements about students.

The influences resulting in specialised procedural knowledge shared commonalities with those behind AWOTS, conceptual knowledge, and procedural knowledge in isolation. The two combinations of influencing elements are:

- Episode goals that emphasise procedures and their solutions or that address student confusion;
- Lesson phases to consolidate content and interactive decisions.

5.3.2.1 Episode goals leading to only procedural knowledge

The analysis of the episode goals leading to procedural knowledge alone revealed two significant trends that were not noted for conceptual knowledge or AWOT episodes. The episode goals for the 57 episodes in which a preservice teacher enacted only procedural knowledge are grouped according to episode goal types in Table 39.

Table 39. Type and relative frequency of episode goals for episodes with only procedural knowledge

Major category of episode goals	Type of episode goal	Total goals for type (n = 174)	Goals for procedural knowledge only		Goals for *specialised* procedural knowledge only	
			No. (n = 69)	Percentage of total goals by type	No. (n = 12)	Percentage of goals for only procedural knowledge by type
Content focused goals	Develop students' knowledge of procedures	60	35	58%	4	11%
	Teach students appropriate use of mathematical language	16	1	6%	1	100%
	Connect procedure with a concept	12	2	17%	0	0%
	Associate procedure with certain types of solutions	10	6	60%	2	33%
	Connect procedure with mathematical purpose	9	0	0%	0	0%
Student focused goals	Address student confusion	32	7	22%	2	29%
	Value and/or encourage student contribution	12	8	67%	3	38%
	Gauge student knowledge	12	6	50%	0	0%
	Avoid student confusion	11	4	36%	0	0%

No trends were evident in Table 39 for episode goal types and the 11 episodes featuring specialised aspects of procedural knowledge in isolation. Where any type of procedural knowledge was concerned, however, Table 39 shows that the episode goal, "Develop students' knowledge of procedures" appeared more often than the other eight categories of episode goals combined. Of the 35 goals of this type, 28 were part of preactive decisions. The strong presence of this episode goal type and its formation in the lesson planning stages reflects the preservice teachers' intentions before and during the lessons to develop particular aspects of their students' procedural knowledge, such as Grace's goal for one episode, "They need to be able to expand." The preservice teachers also tended to teach only procedural knowledge when they wanted to point out to their students the kinds of solutions they should expect when performing algebraic procedures. For example, Grace told her students in one episode that the solutions to a set of simultaneous equations would be "yucky answers" because the solutions were not integers. In her reflection, she commented, "That was just to warn them more than anything that, 'If this turns out badly, it's not that we've made a mistake.'" This tendency is shown in Table 39 in which 60% of goals to "Associate procedure with certain types of solutions" resulted in only procedural knowledge being enacted.

The episode goals were located within broader lesson goals that also emphasised algebraic procedures. Of the 19 lesson goals outlined in Table 13 (section 4.2.1), 18 referred to the preservice teachers' intent to develop their students' proficiency in particular algebraic procedures, which was consistent with the procedural emphasis that existed throughout the observed lessons. The lesson goals were aligned with the term overview content so the practicum setting is implicated in the procedural goals established. The priority the participants gave to mastering algebraic procedures rather than developing other types of mathematical knowledge in their lesson goals tended to result in episodes featuring procedural steps, rules, and facts without conceptual or AWOT connections.

Preservice teachers also tended to present only procedural knowledge when they were paying close attention to their students' mathematical understandings. Table 39 shows that when the participants were looking to encourage their students' mathematical contributions, only procedural knowledge tended to manifest. The table also shows that half of the episode goals to gauge student knowledge also led to enacted procedural knowledge in isolation. These statistics may be a reflection of the lesson goals focusing so heavily on procedural mastery, which may have led to preservice teachers looking no further, where their students' understandings were concerned, than their ability to get the right answer.

5.3.2.2 Only procedural knowledge: Preactive decisions, impacted by judgements about students

In contrast to AWOT and conceptual knowledge episodes, procedural knowledge was enacted alone as the result of pre-planned decisions in every lesson phase type. Table 40 provides a summary of the episodes that only manifested procedural knowledge, according to lesson phase type and whether classroom events prompted the related decisions.

Table 40. Procedural knowledge episodes by lesson phase and classroom event

Lesson phase	Total episodes for type	Only procedural knowledge episodes		Only procedural knowledge episodes by classroom event	
		No.	Percentage of total episodes for type	No event	Classroom event
Introduce content	17	3	18%	2	1
Consolidate content	43	18	42%	11	7
Develop content	20	9	45%	8	1
Review content	57	27	47%	20	7
Total	137	57	42%	41	16

Table 40 shows that preactive decisions, i.e., that were not prompted by a classroom event, led more often to procedural knowledge alone in every lesson phase. This included the consolidation phase, where the participants were found to have enacted conceptual knowledge and AWOTS more often in response to classroom events than as a result of their lesson image. Preactive decisions are those that the preservice teachers have more time to make because they are established prior to the lesson. It would be reasonable to expect that with more time to consider the lesson content and to decide upon aspects of that content that require particular emphasis, preactive decisions would regularly feature conceptual knowledge and AWOTS but this was not the case. Enacting only procedural knowledge was a pre-planned choice in almost half of the episodes resulting from preactive decisions in this study (41 out of the 92 episodes resulting from preactive decisions overall). An analysis of the preactive decisions that led to the 41 episodes, highlighted in Table 40, revealed the influence of the preservice teachers' judgements about students on those preactive decisions.

The participants offered rationales for their preactive decisions in their reflections of every episode where procedural knowledge was enacted alone (n = 41). The rationales were part of additional explanations the preservice teachers chose to provide in their interviews. In the rationales they provided for the 41 episodes based on preactive decisions, they referred to the judgements they held about students for 36 of those episodes. The dominant line of thought articulated by all preservice teachers concerned the belief that for students to fully master an algebraic procedure, they needed repeated exposure to the procedure, with clearly laid out steps. Kate, for example, explained that procedures were "something that you just need to practise and you get better." This may suggest that preservice teachers are not aware that relational understanding of procedures can only be developed by students when connections with conceptual knowledge are emphasised (Kilpatrick et al., 2001; Skemp, 1979). Alternatively, the preservice teachers may hold judgements of this kind but other beliefs they hold about students that relate to regular practice and following sequences of steps are more influential when they plan their lessons. In either case, the end result was conceptually impoverished planning and teaching actions.

5.3.2.3 Episode goals leading to specialised procedural knowledge

Two types of goals comprise the majority of episode goals leading to specialised procedural knowledge. Table 41, which summarises the episode goals associated with specialised procedural knowledge, shows the episode goal, "Develop students' knowledge of procedures," featured in the preservice teachers' decisions to present specialised procedural knowledge just as it did for procedural knowledge in isolation.

Table 41. Type and relative frequency of episode goals for specialised procedural knowledge episodes

Major category of episode goals	Type of episode goal	Total goals for type (n = 174)	Goals for specialised procedural knowledge No. (n = 42)	Percentage of total goals by type
Content focused goals	Develop students' knowledge of procedures	60	17	27%
	Teach students appropriate use of mathematical language	16	2	13%
	Connect procedure with a concept	12	1	8%
	Associate procedure with certain types of solutions	10	3	30%
	Connect procedure with mathematical purpose	9	2	22%
Student focused goals	Address student confusion	32	11	34%
	Value and/or encourage student contribution	12	4	33%
	Gauge student knowledge	12	0	0%
	Avoid student confusion	11	2	18%

Table 41 also shows that specialised procedural knowledge manifested in 11 episodes when preservice teachers addressed student confusion. Of those 11 episodes, two also featured AWOTS and seven others featured both enacted ways of thinking and conceptual knowledge. Hence, it appears that when preservice teachers think about the troubles their students are experiencing, they tend to teach the better aspects of their MCK, including AWOTS, conceptual knowledge, and specialised procedural knowledge.

5.3.2.4 Specialised procedural knowledge: Lesson phases to consolidate content and interactive decisions

Student generated events in consolidation phases were associated with specialised procedural knowledge, just as they were for AWOTS and conceptual knowledge. An overview of specialised knowledge episodes, organised according to lesson phases and the presence of an influencing classroom event, is presented in Table 42.

Table 42. Specialised procedural knowledge episodes by lesson phase and classroom event

Lesson phase	Total episodes for type	Specialised procedural knowledge episodes		Specialised procedural knowledge episodes by classroom event	
		No.	Percentage of total episodes for type	No event	Classroom event
Introduce content	17	3	18%	3	0
Consolidate content	43	14	33%	5	9
Develop content	20	6	30%	6	0
Review content	57	12	21%	7	5
Total	137	35	26%	21	14

Table 42 shows that specialised procedural knowledge episodes were more often the result of interactive decisions in phases where preservice teachers were aiming to consolidate content. Interactive decisions also led to just under half of the episodes in review content phases. Although the preservice teachers chose to deliver specialised knowledge of procedures in response to student contributions in these phases, they were less enthusiastic about doing so compared with the AWOT and conceptual knowledge episodes discussed in sections 5.1 and 5.2.

Competing influences, reported in detail in chapter 4, were mentioned by all participants when they reflected on the specialised procedural knowledge they considered sharing with their students. Five of the six participants (all but William) reflected in their interviews that during their lesson, they considered taking a modified solution path when one of their students suggested they do so. In some instances, they decided not to take up their students' suggestions, intentionally omitting specialised procedural knowledge, which is discussed in section 5.3.3. Only three preservice teachers, Kate, Sam, and Ben chose to communicate their knowledge of an alternate solution path when prompted by a student suggestion and modify the method they had originally intended to share with their students before the lesson began.

The three preservice teachers privileged their students' mathematical preferences over the approach they had expected was most suitable before the lesson, reflecting a change in the judgements they held about which solution paths would best suit their students. Each offered a goal that related to acknowledging their students' mathematical ideas. Kate and Sam seemed relatively more comfortable obliging their students' requests. Kate, for example, explained, "When they suggest things, I like to go with it, 'cause if they're talking me through, it is better than me talking them through it. Like, they're telling me what to do so it shows me they're sort of getting it." Ben was less flexible with his approach to student suggestions. Although he recognised when a solution path suggested was mathematically sound, he still insisted on what he perceived to be a better solution path. When Ben did explore alternate solution paths, he did so begrudgingly to appease his students, as the example below illustrates.

Having given his students time to solve the equation, $\frac{4(x+7)}{3} = 4$, Ben questioned his students about how they should begin solving. The response he was looking for was one where the expressions on either side of the equation would be divided by four. He tried to prompt students for this idea, but with no success, and in the end, was resigned to performing the procedure "their way." His eventual change of heart is evident in the following episode excerpt.

Ben: Isolate. Correct. We have to isolate. Now, how are we going to isolate this? What are we going to use?

[Some students call out “opposite operations,” while others say “claw method.”]

Ben: Yup. Oh, the claw method? We could use that but opposite operations is correct. So the claw method can be used, the distributive law, but we’re going to use instead, we can use opposite operations. What’s our first operation here?

[Long pause, no-one responds.]

Ben: What are we going to do? What are we trying to get rid of first?

Student: Maybe we can do, expand brackets?

Ben: Yeah, we can expand the brackets…

[Ben waits but no-one makes any other suggestions.]

Ben: Yeah, we’ll expand the brackets. That’s fine. Four x plus 28. Not a bad tactic to get into.

As Ben watched the footage back in the interview, he laughed, remarking, “Stuff doin’ that? Fine, I give up!” Ben’s desire to pursue a different solution path was evident in his interview, as he explained, “It’s more complicated if they expand the brackets. Because three times four, divided by four, is a lot easier than you know, four x plus 28 divide, and then, equals 12.” Ben felt that dividing both expressions by four would produce the most efficient solution path but was still able to draw on his specialised knowledge and improvise an adjusted set of steps that aligned with how his students suggested the equation might be solved.

In this episode, Ben managed a number of decision making influences as he decided how to respond to his students. Ben weighed up his own mathematical preference (Influence 4, preservice teachers’ MCK), his changing opinion about which method would be most beneficial for his students to see (Influence 5, judgements about students), and his desire to acknowledge the ideas of his students (Influence 2, micro goals of the lesson) all in a matter of moments (Influence 3, student prompted interactive decision). In this episode, the attention Ben paid to his students' opinions was instrumental in shaping the MCK that he decided to teach. Live student-teacher interactions can therefore alter preservice teachers' pre-existing plans to enact particular MCK if they possess the specialised procedural knowledge to do so and if they are prepared to prioritise student suggestions over their own intended method.

5.3.3 Limitations in the procedural knowledge enacted

The preservice teachers occasionally communicated their procedural knowledge in ways that distorted mathematical meaning within the lesson and restricted its application beyond the lesson context. The analysis of the procedural knowledge presented by the preservice teachers in the 125 episodes revealed three limitations. First, the preservice teachers were again imprecise with their verbal descriptions of procedures. Second, the preservice teachers presented their procedural knowledge in automated chunks, failing to attend to different structural features of examples or to note the limitations of the procedures they presented if they were applied beyond the mathematical context of the lesson. Third, the preservice teachers intentionally withheld valuable aspects of their procedural knowledge from their students. Elaborations of each limitation show how the preservice teachers' actions hindered their students' opportunities to develop a fuller understanding of algebraic procedures.

Imprecise language to describe algebraic procedures was used regularly by the preservice teachers. Verbally imprecise descriptions of procedures, concepts, or ways of thinking were evident in 70 of the 125 episodes (56%) where procedural knowledge manifested. It is likely that the participants' imprecise forms of communication made mathematical meaning more difficult for their students to fully grasp. Table 43 shows the presence of each type of imprecision across those 70 episodes. The total percentage exceeds 100% because up to four types of imprecision were present within an individual episode.

Five of the six preservice teachers regularly used mathematical or non-mathematical language regarding procedures in an inappropriate mathematical context (see Table 43), which can distort the clarity of the mathematics presented (Heaton, 1992; Hill, Blunk et al., 2008).

Table 43. Types of verbal imprecision in episodes featuring procedural knowledge

Type of verbal imprecision	Episodes (n = 70)
Language used in incorrect mathematical context	42 (60%)
Language features non-mathematical, informal code	32 (46%)
Language is ambiguous	16 (23%)
Language is overly casual	9 (13%)
Language forms a maze - incoherent tangle of words	4 (6%)

Grace, for example, apologised to her students for creating two equations that, when solved simultaneously, produced "yucky answers." Grace was, in fact, referring to the presence of fractions in the solution to an algebraic procedure but her students had no way of knowing what mathematical notion she was talking about.

Thomas used language that would normally be found in an algebraic context but he did so imprecisely, when he asked his students first, to "solve" the expression, $x + 23$ when $x = 9$, and second, to "find the value of x" for the equation, $2x - 10 = 15$. Thomas' use of the word "solve" rather than a more appropriate direction, such as evaluate, for his first task and the absence of the word solve for the second task, reduced the mathematical clarity surrounding algebraic language (i.e., "Solve") and associated algebraic objects (i.e., equations, not expressions). Thomas' imprecise use of algebraic language, like the other preservice teachers when they produced similar verbal distortions, limited the overall quality of the procedural knowledge enacted.

Other types of verbally imprecise references to procedures included coded or invented language that preservice teachers used to describe certain procedures, casual descriptions of procedural steps, and mazes. Statements such as "off they go," used by Ben when he referred to a cancelling procedure for the expression, $\frac{x}{6} \times 6$, failed to capture the mathematical procedures being presented with accuracy. In Ben's case, his description of the cancelling procedure, according to Falle (2005), could potentially contribute to students developing an unclear and inaccurate understanding of the procedure. Language such as "timesing" (William) or "minusing" (Thomas) that is mathematically careless or sloppy and reduces the mathematical quality of explanations (Rowland et al., 2011; Sleep & Eskelson, 2012), were also regularly employed by the preservice teachers. Procedural mazes or tangles of words were present in episodes like Sam's, when, without any written notes, he offered the following explanation to two students about operating with integers: "So if we're multiplying and dividing by positives, we're going to have positives, negatives, we'll end up with positives." Not only were the preservice teachers' students required to make sense of the mathematics content being presented but they were also faced with interpreting loosely expressed procedural language. Hence, imprecise references to procedural steps and rules regularly diminished the mathematical quality of the episodes.

A one-size-fits-all, automated approach to algebraic procedures was presented and implicitly encouraged by all preservice teachers, limiting the procedural knowledge enacted. The notion of unpacking mathematical content to make aspects of that content more accessible to students (Ball et al., 2008; Kilpatrick et al., 2001; Ma, 1999) did not always apply to the preservice teachers' actions. Procedural steps were articulated by all preservice teachers at least once in their lessons that were either (a) implied as absolute, despite other procedural options being available or (b) unnecessary, given the mathematical structure of the question. Verbal descriptions of the steps intimated that they should be performed automatically and without consideration of the necessity or merit of the actions taken. One of William's episodes captures both kinds of automation that were present at different points in the lessons of all preservice teachers.

Automation was present throughout William's episode involving the warm up question, "Expand: $-5\ (3d - e)$." The first automated step used to manipulate the expression was captured by a "trick" (see Appendix R) that William used repeatedly in his review phase with slight variations, "Multiply through, everything outside times everything inside." What William failed to make clear was whether he was using the distributive law of multiplication over addition (with the additive inverse concept) or subtraction to perform the procedure. He had instead condensed the two possible solution paths into one absolute procedural catchphrase that according to William could be applied without paying further attention to the operation within the brackets.

William's overall procedural approach for the example was also automated because the completion of his first step immediately triggered an additional, but unnecessary second step. The removal of the brackets from the expression led to an immediate "next step," which was to simplify the expression by collecting like terms. In this case, the search for like terms was pointless because the structure of the first expression showed that no like terms would exist in the second, equivalent expression. William did not mention in the interview whether he was aware that the final step was redundant but his repeated requests that his students look for like terms in later questions seems to indicate that the two steps should be automatically performed as far as William was concerned. Once more, William paid no attention to the structure of the expression that he was manipulating and instead focused on only one automated sequence of rules and steps.

Similar instances of automated rules and the application of those rules "every time" were identified in all preservice teachers' lessons. This suggests that preservice teachers sometimes choose to expose their students to a far narrower view of algebraic procedural options than what might be desirable to help students become flexible with their use of procedures in different algebraic contexts. When the preservice teachers spoke of solving equations, the limitations associated with the procedural rules and sequences of steps they taught, if extended to more advanced equation structures, were not alluded to by the preservice teachers. They also failed to tell their students that the steps they outlined would produce only one (relatively efficient) potential solution path but other paths were also possible. Rather than allowing students the opportunity to find more equivalent equations or multiple solution paths that shared common conceptual principles, the preservice teachers potentially distorted their students' perspective on how equations can be solved, reducing the quality of the procedural knowledge they shared with their students.

The analysis of the nine episodes with these types of limitations revealed that the majority (eight episodes) were the result of preactive decisions. The mathematical distortion created was therefore not due to the preservice teachers not having time to reflect on the applicability of the procedural knowledge they taught to broader mathematical contexts beyond those in the lesson. The eight episodes also took place within whole of class instructional settings so the restriction the preservice teachers placed on the procedural knowledge they shared was not one that the teachers deemed necessary only for a particular student but one they felt suited the whole class. It appeared that short term success with procedures within the lesson was privileged by the participants who restricted, intentionally and unintentionally, the transferability of the procedural knowledge they taught beyond the observed lessons.

The third limitation concerned the preservice teachers' intentional omissions of procedural knowledge. All participants, with the exception of Kate, commented in their interviews on teaching actions which they decided not to undertake. This occurred 25 times during the preservice teachers' reflections of 22 episodes. Each time, the preservice teachers described procedural knowledge which they considered enacting but eventually chose to withhold from their students. The procedural knowledge that the preservice teachers chose to omit was organised into three categories.

Firstly, preservice teachers decided to refrain from enacting "more of the same" procedural knowledge either before or after ten of the episodes. They did so by choosing not to show their students how to complete a particular problem or failing to complete a procedure for the class. Grace, for example, decided at two different points in her first lesson not to model the latter stages of the substitution method for solving simultaneous equations and instead, moved on to new examples. Her reasons for doing so are described in section 5.3.4. The effect of these kinds of omissions was not that different knowledge was missed by the preservice teachers but rather that repeated exposure to certain procedural actions introduced in previous lessons was reduced. Although omissions of this type did not always limit the procedural knowledge enacted, in Grace's case, her students were not shown a complete solution of the substitution method, a new method for the students, at any point in the lesson. This limited her students' opportunities to recognise the similarities between solving simultaneous equations using the substitution and elimination methods. Therefore, intentional withholding of what is considered "more of the same" MCK can, but does not always, limit the MCK enacted.

Secondly, four of the preservice teachers saw nine opportunities to enact alternate procedural knowledge that they had not originally planned to mention, but decided against enacting that knowledge. For example, in one of Sam's lessons, he had just finished solving a linear equation using firstly, the backtracking method, and secondly, the transposing method. A curious student began the following conversation about the number of methods that existed to solve a linear equation, which Sam initially seemed interested in pursuing but then changed his mind and quickly ended.

Student: Are these, like, two different ways you can do it? Just two?

Sam: Oh, no, there's lots of different ways you can do it, if you want. You could just guess numbers for x and put them in there. There's lots of different ways you can do it.

[The class murmured with interest as Sam looked to the back of the room at his supervising teacher].

Sam: Don't do it, but you could. It'll take you a very long time.

As Sam watched the video footage of the excerpt above in the interview, his reflection revealed that he had additional knowledge of an alternate solution path but he had decided not to tell his students about it. The reasons Sam and the other participants gave for withholding their knowledge of additional solution paths are reported in section 5.3.4.

Thirdly, the preservice teachers decided on six occasions not to enact particular specialised knowledge that they possessed about features of algebraic procedures. Four preservice teachers, William, Sam, Ben, and Thomas chose to consciously refrain from highlighting a particular feature of an algebraic object or procedure. Sam, for example, presented the equation, $26 = 3z + 5$ to his class to solve. This equation was different in structure from the class's previous examples because two operations had been performed on the unknown, rather than only one. Sam did not point this difference out to his students. He also didn't explicitly mention that the method needed to solve the equation was therefore slightly different from the method his students had relied upon previously. He remarked in the interview that he was aware at the time that the structure of the example and the subsequent procedure were more complex but chose not to tell the students. Later in the lesson, confusion amongst the students and a chat with his supervising teacher confirmed that it would have been valuable for Sam to enact his knowledge about features of examples and related procedures. Similar omissions from three other preservice teachers showed that even if their MCK is unpacked, they may decide not to address procedural features with students. Their view contrasts with the literature on specialised content knowledge which advocates making features explicit to students (Ma, 1999; Kilpatrick et al., 2001).

5.3.4 Possible influences producing limitations in the procedural knowledge enacted

The researcher inferred two possible influences leading to verbally imprecise or automated procedural knowledge, as the participants rarely reflected specifically on those two limitations. For intentional omissions, however, the preservice teachers referred to competing influences in their reflections. The influences behind the preservice teachers' verbal imprecision and automated treatment of procedures are elaborated first, followed by an examination of the competing influences behind the omitted procedural knowledge.

Poor language choice appeared to be the result of underdeveloped MCK and judgements about students. The language preservice teachers used to explain their actions in the interviews usually featured the same imprecise language they used in the episodes, rarely indicating that they possessed stronger verbal descriptions of MCK than those they shared with their students.

On the few occasions where stronger verbal language was noted in the interviews, the preservice teachers' student judgements appeared to undermine their attempts to provide high quality procedural explanations. Very occasionally, Sam and William spoke more precisely in their episode reflections and when the researcher pointed out that they were using different language, they drew upon their judgements about students to justify their weaker language choices in class. For example, Sam intentionally used the word "put" instead of the word "substitute" when he told his students to "put" numbers into an equation. In his interview, he reflected, "I'd definitely normally use the word substitute. Maybe not with Grade 8. I've run into trouble with that before. They weren't sure what I meant, so I said, 'Put it into the equation.'" For Sam and William, it appeared that avoiding confusion, a noble pedagogical goal in many instances, justified their decision to use language with less precise mathematical meaning.

The preservice teachers' reflections indicated that they were very comfortable with the mathematical language they used with students, suggesting their judgements about the mathematical language that students should be exposed to were not ideal. In some cases, preservice teachers even chose to mimic their students' poor language choices during student interactions. Only two of those preservice teachers, William and Ben, acknowledged their poor language choice at one point in the interview, reflecting on their difficulty in finding the right words to use for such inexperienced algebra students. Reflections such as these were found very rarely. The findings suggest preservice teachers' classroom experiences may not encourage them to develop more precise language but may instead encourage them to stay within a less precise and less meaningful verbal comfort zone. The value of knowing and communicating precise algebraic language to enhance students' understanding of the content was not recognised by any of the preservice teachers in the interviews and verbally impoverished episodes were the result.

The automated treatment of mathematical procedures by the preservice teachers appeared to be influenced most strongly by their judgements about students. Discussions of algebraic procedures in the interviews by all preservice teachers evidenced a more flexible approach to procedures than what they were prepared to teach their students. Within those same discussions, the preservice teachers implied that the automated steps they encouraged during instruction would result in their students making fewer errors and therefore experiencing more immediate success in the lesson. The automatic decision to "expand the brackets" whenever an expression involving brackets is encountered was encouraged to various degrees by all preservice teachers in their lessons and the perceived simplicity that accompanies an automated rule like this one illustrates the apparent appeal of presenting mathematics as automated sets of procedures or routines to students.

Expanding brackets whenever they were encountered was explicitly encouraged by Grace, strongly implied by William, Sam, Kate, and Thomas and tolerated by Ben. Grace encouraged her students to, "Expand these brackets," even if the multiplier was one and the use of the distributive law was unnecessary (e.g., simplification of the expression, 5y + (5 + 2y)). She encouraged her students to write "1 ×" in front of any brackets and expand the expression using the distributive law, justifying her actions in the interview by saying, "They've already got so many steps. I'd prefer to give them, 'This is how it works for everything,' rather than, 'For this case here we can just get rid of the brackets.'" Ben was aware that his students appeared to automatically expand any expression with a bracket without paying more attention to the overall structure of an expression or equation but seemed unsure if broadening his students' procedural horizons was a good idea. As discussed in section 5.3.2.4, when Ben's students argued to expand the expression in the numerator in the equation, $\frac{4(x+7)}{3} = 4$, as an automatic first step, Ben prompted for other suggestions for a few moments, then eventually relented and followed his students' requests. He laughed as he reflected in his interview,

> It's probably good for them to get into the habit of just seeing a bracket and going, 'Oh, expand, expand, expand!' [laughing]. Rather than asking questions, 'Just expand!' Rather than [thinking], 'Hey maybe we can do this easier!' [laughing]. Just go with it.

The perceived benefit of students being better able to use one set of steps for a variety of situations in the limited context of their current algebra unit was prioritised by the preservice teachers over the more global aim of developing a flexible approach to performing mathematics procedures. Ben and Sam were the only preservice teachers to acknowledge that they might be sending the wrong mathematical message to their students but neither regretted their actions, despite their admissions. Sam, for example, insisted in his lesson that his students solve all linear equations by undoing operations performed on a pronumeral in the order specified by his "SADMOB" rule. In the interview, as Sam talked with the researcher about his approach, the researcher wrote down the equation, $\frac{2(x+7)}{3} = 18$, on a piece of paper and asked Sam how he would go about using his SADMOB rule to solve it. He laughed, saying, "I think I might be in trouble," as he mused, "Hmm, but why would you do three first? Why would you necessarily bring three up first? I guess you'd just teach that separately." Sam acknowledged the limitations of his rule and demonstrated an alternative approach to solving equations in the interview, admitting that he didn't rely on the "SADMOB" rule himself when solving. Nevertheless, Sam still held fast to his rule, insisting, "I think for kids, it's the best way." The success of the other preservice teachers at tertiary level mathematics and their interview reflections suggest that they, too, knew a range of approaches to solving equations, but, like Sam, were choosing instead to deliver a limited version of their MCK for the "good of their students."

Intentionally omitted procedural knowledge was the product of four of the influences reported in chapter 4: the practicum context, discarded goals, the preservice teachers' MCK, and their judgements about students. Competing sets of influences that resulted in procedural knowledge being withheld are summarised in Table 44 and capture the complex nature of the participants' decisions when they were less certain of which MCK path to take.

Table 44. Influences prioritised over discarded goals

Influence(s) prioritised by preservice teacher	Discarded goals	
	Content focused	Student focused
Influence 1: Practicum context	2 (11%)	-
Influence 4: Preservice teachers' MCK	2 (11%)	-
Influence 5: Preservice teachers' judgements about students	4 (22%)	2 (29%)
Influences 1 and 4: Practicum context, preservice teachers' MCK	4 (22%)	2 (29%)
Influences 1 and 5: Practicum content, preservice teachers' judgements about students	3 (17%)	1 (14%)
Influences 4 and 5: Preservice teachers' MCK and judgements about students	3 (17%)	2 (29%)
Total (n = 25)	18 (72%)	7 (28%)

Table 44 shows that the practicum context features strongly in the prioritised influences, overriding 12 of the 25 discarded goals that the participants considered pursuing but chose to omit. The preservice teachers omitted procedural knowledge to ensure that lesson content would be covered in time, reflecting the influence of the term overview and their supervising teachers' advice. The preservice teachers also omitted procedural knowledge as a result of direct advice from their highly respected supervisors. Sam, as described in section 5.3.3, briefly began, but then abruptly ended a conversation with his class about an alternate method for solving equations. In his interview, Sam laughed at the footage showing him changing his mind, saying,

> I meant what I said. You could definitely try numbers for x. It's a valid mathematical way of doing it… That's something you do, later on, definitely. If it's too hard to solve, analytically, you definitely use trial and error. You run it through a program and use trial and error… And then Jenny [supervising teacher] started shaking her head and I thought, "Oh, I better not say that."

Hence, the practicum context impacted the preservice teachers' choice of procedural knowledge to omit, just as it also influenced their choice of procedures to deliver.

The participants' judgements about students also featured heavily in the prioritised influences in Table 44. Of the 25 times that preservice teachers omitted procedural knowledge, their judgements about students were prioritised 15 times. Additional analysis of the related episodes revealed that those judgements pertained to large cohorts of students on 14 of those 15 occasions because the associated instructional settings involved conversations conducted in front of the whole class. Each preservice teacher who omitted procedural knowledge did so one or more times to avoid confusing their class, as Grace's example illustrates.

Grace appeared, at one point in her lesson, to brush off a student's suggestion to take a different but mathematically sound solution path when rearranging a linear equation to isolate a pronumeral. In the interview, Grace reflected that she would have happily followed along with her student's suggestion but was worried the rest of the class would become confused if she deviated from her standard way of performing the procedure. She reflected, "They might get confused if I say something different." Grace's and the other preservice teachers' reliance on similar judgements led to lost opportunities to demonstrate different approaches to performing procedures. Had these other solution paths been enacted, students could have been exposed to a wider variety of possible procedures and may have had the opportunity to discuss with their teacher the possibility of taking different options when solving and judging the merits of those options for the question at hand. For the preservice teachers, their judgements about students led them to limit the procedures enacted to those that were perceived as absolutely essential, rather than to present a broader range of procedural options. In the preservice teachers' opinions, limiting procedural options by omitting certain MCK was what they felt they needed to do to adequately meet their students' needs within their practicum lessons.

5.4 Conclusion

The participants of this study enacted three types of MCK: AWOTS, conceptual knowledge, and procedural knowledge. A summary of the MCK enacted by the participants of this study is provided in Table 45. Enacted horizon knowledge is included in the table but was not reported in this chapter because the majority of the participants did not enact this type of MCK.

Table 45. Summary of MCK types enacted in episodes

Type of MCK	Episodes, by MCK sub-domain		
	Common content knowledge	Specialised content knowledge	Horizon knowledge
Algebraic ways of thinking	67	11	-
Conceptual knowledge	48	18	-
Procedural knowledge	125	35	1

Table 45 shows AWOTS and conceptual knowledge were enacted by the preservice teachers intermittently and as discussed previously in this chapter, usually without a great deal of depth. In contrast, procedural knowledge was enacted in the majority of episodes, appearing to be the preservice teachers' mathematical comfort zone in the classroom. Presenting a logical sequence of mathematical steps is not necessarily problematic but the emphasis that the preservice teachers gave to automated procedures at the exclusion of other equally valuable types of mathematical knowledge is troubling. Specialised types of MCK were evident in certain episodes, suggesting that preservice teachers are beginning to develop specialised kinds of MCK for their work as teachers. However, the MCK communicated was often verbally imprecise and the participants failed to identify the limitations of the procedures they presented in class, if applied to more advanced mathematical contexts.

The influencing elements described in chapter 4 lay behind different types and qualities of MCK. Tables 46 and 47 show summaries of the influences that tended to impact most significantly the participants' decisions to enact a certain type of MCK (Table 46) and limited versions of each MCK type (Table 47).

Table 46. Summary of influences impacting decisions to enact MCK types

Influence	AWOTS	Conceptual knowledge	Procedural knowledge
Practicum context			✓
Preservice teachers' goals	✓	✓	✓
Live classroom circumstances	✓	✓	✓
Preservice teachers' MCK	✓		✓
Preservice teachers' judgements about students	✓		✓

Table 47. Summary of influences impacting decisions to enact limited MCK types

Influence	AWOTS	Conceptual knowledge	Procedural knowledge
Practicum context			✓
Preservice teachers' goals			✓
Live classroom circumstances			✓
Preservice teachers' MCK	✓	✓	✓
Preservice teachers' judgements about students		✓	✓

The tables show that all five influences impacted the participants' decisions to enact procedural knowledge with and without limitations. Lesson goals, episode goals, and judgements about students that prioritised mastery of procedures were particularly influential and led to procedurally dominated teaching actions. Enacted conceptual knowledge and AWOTS were not strongly associated with elements of the practicum context but were closely related to the remaining four influences. The influence of classroom circumstances on MCK related decisions is particularly significant because student generated classroom events tended to result in the preservice teachers enacting the better aspects of their MCK, namely, conceptual knowledge, AWOTS, and specialised procedural knowledge. Spontaneous MCK related decisions, prompted by student contributions, appeared to lead preservice teachers to rethink the content they were delivering and may have helped them to unpack the MCK they held in a tacit form. Situations of this type tended to occur when preservice teachers were consolidating content with their students, often in small instructional settings.

Underdeveloped MCK and judgements about students appeared to be significant contributing influences when preservice teachers presented limited MCK. The preservice teachers' underdeveloped references to conceptual ideas and AWOTS suggest that they are unable to access pertinent conceptual knowledge or ways of thinking to deal effectively with all classroom situations. Although tacit MCK could be evidenced in the preservice teachers' instructional actions, the current form of their MCK, whether incomplete or compressed or both, meant that they were not able to draw more successfully on the conceptual knowledge or ways of thinking needed in their teaching practice, missing numerous opportunities to teach mathematics content more deeply.

Interestingly, no preservice teachers noted that their supervising teachers commented on the limitations described in this chapter and for the most part, appeared unaware that their presentation of lesson content was limited in any way.

The findings of this chapter highlight the unique circumstances that preservice teachers encounter in live classroom contexts that lead them to enact particular aspects of their MCK. Chapter 6 synthesises the findings of this chapter and the previous chapter and provides a discussion of the synthesised findings with respect to the literature concerning MCK, teacher decision making, and the education of secondary mathematics preservice teachers.

Chapter 6: Synthesis & Discussion

6.0 Introduction

This study examined, from a situated knowledge viewpoint, the mathematical content knowledge (MCK) of algebra that preservice teachers decide to enact in the classroom. Concerns have been raised about the mathematical capabilities of secondary mathematics preservice teachers and decontextualised measures of preservice teachers' MCK taken from responses to written assessment items and/or interviews have highlighted inadequacies across a number of mathematical topics (e.g., Bryan, 1999; Even, 1993; Goos, 2013). However, little is known about how secondary mathematics preservice teachers draw on or develop their MCK during a practicum lesson.

This study was consequently undertaken with the intent of studying preservice teachers' MCK related thoughts and actions as they taught lower secondary algebra classes. More specifically, the aims of the study included gaining a better understanding of the MCK that preservice teachers decide to enact while teaching a lower secondary algebra lesson (research question 2) and of the elements that influence their MCK related decisions (research question 1). The classroom actions of the preservice teachers and their post-lesson reflections provided insights into the MCK that manifested in their actions (chapter 5) and the decision making thoughts behind those actions (chapters 4 and 5). This chapter continues to draw together the findings pertaining to the two research questions.

The first section of the chapter synthesises the findings of this study. For purposes of comparison, the major findings are described and examined in light of the literature. The second section, consistent with an interpretivist approach, again draws on the literature but this time for the purpose of offering possible explanations for the kinds of MCK related decisions that the preservice teachers made and the quality of the enacted MCK that ensued. This section provides additional insights into the experience of the preservice teachers when they enact MCK in the practicum classroom.

6.1 Synthesis of findings

This chapter section presents the major findings of the study in two parts. Firstly, the visible MCK that preservice teachers chose to enact in their lessons is summarised and compared with the findings of other studies of preservice teacher MCK. Secondly, a synthesis of the elements that influence the invisible aspects of enacting MCK, that is, the preservice teachers' MCK related decisions, is provided. This synthesis also includes conclusions about the impact of particular influencing elements on the quality of MCK enacted.

6.1.1 Major findings addressing research question 2

The second research question asked "What is the mathematical content knowledge (MCK) that secondary preservice teachers enact when teaching lower secondary algebra?" Overall, the preservice teachers presented algebraic content that emphasised mastery of procedures, reflecting a highly procedural and automated approach to algebra. As they did so, problematic aspects of their own MCK were exposed alongside more favourable aspects. This chapter section presents a discussion of the four key findings that address research question 2. The preponderance of procedural knowledge in the participants' teaching actions is first examined with respect to the literature. Next, a discussion of the two less than desirable MCK qualities, the limited presence of conceptual knowledge and algebraic ways of thinking (AWOTS) and the verbal imprecision of mathematical ideas, are presented. The final key finding in this section concerns the emerging presence of aspects of the participants' MCK that are needed for teaching.

6.1.1.1 A preponderance of enacted procedural knowledge

The most prominent type of MCK enacted by the participants in this study was procedural knowledge, evidenced in 91% of the episodes observed. The preservice teachers regularly paid explicit attention to mathematical rules, facts, and steps associated with the algebraic procedures they presented to their students. Any conceptual knowledge or AWOTS presented by the participants were usually in support of their delivery of procedural knowledge which was the common thread running through almost every episode and sequencing most episodes from one to the next. An approach to mathematics teaching that concentrates excessively on procedures is considered an impoverished one by many scholars (Eisenhart et al., 1993; Hiebert & Le Fevre, 1986; Kilpatrick et al., 2001; Rowland et al., 2009; Skemp, 1979). The finding suggests that preservice teachers spend a disproportionate amount of their lessons showing students how to "get through the steps" and "get the right answer" rather than investing time developing other valuable aspects of students' mathematical understandings.

The preservice teachers' overemphasis on procedural knowledge is consistent with the findings of other studies of preservice teachers' instructional actions in the practicum and university setting. Livingston and Borko (1990) studied two secondary preservice teachers' practicum lessons about calculus and analytic geometry and concluded that the preservice teachers' instructional explanations and questions were both procedural in nature. The preservice teachers compared poorly with the two expert teachers in the same study who were able to draw their students' attention to concepts as well as procedures in their lessons. In the university setting, Plummer and Peterson (2009) investigated a secondary preservice teacher's mock mathematics lesson about operations with integers that she presented to her peers as part of a mathematics education course. The participant's teaching actions also revealed a preference for enacting only procedural knowledge, despite the statements she made during the mathematics education course that emphasised the importance of mathematical concepts. The consistency of findings between this study and the studies of Livingston and Borko (1990) and Plummer and Peterson (2009) suggests that enacting procedural knowledge may be the pedagogical comfort zone for secondary mathematics preservice teachers.

The version of procedural knowledge that the participants in this study presented to their students was a curtailed version of the procedural knowledge that they themselves held of lower secondary algebraic procedures. Notably, the procedural knowledge enacted in this study was knowingly delimited by the preservice teachers in two ways: (a) the intentional omission of procedural knowledge and (b) the intentional restriction of procedural knowledge.

In comparison with their unawareness of notable omissions of conceptual knowledge or AWOTS, the preservice teachers omitted pedagogical knowledge in an intentional way. Five of the six preservice teachers admitted to intentionally withholding additional aspects of procedural knowledge. In doing so, the preservice teachers tended to present a less flexible approach to algebraic procedures than what they held themselves. A flexible approach to procedures is considered advantageous for teachers to possess (Ball et al., 2008; Star & Stylianides, 2013) and in this study, the participants did appear in the interview to possess a more flexible approach to performing lower secondary algebraic procedures than what they demonstrated in their lessons. However, for secondary students to benefit from a teachers' flexible procedural knowledge, that knowledge must be explicitly taught and not withheld. Although the preservice teachers' intentions for withholding a more flexible approach to algebra appeared to be attentive to student needs (discussed in section 6.1.2), their decisions ultimately reduced the quality of the mathematical content that they delivered.

The findings of this study extend the findings of previous research of preservice teachers' live classroom actions by investigating the MCK that was intentionally withheld. Of the five studies that were reviewed in chapter 2 pertaining to secondary mathematics preservice teachers' classroom actions, only one study (Borko & Livingston, 1989) notes the intentional withholding of MCK. Borko and Livingston (1989) found in their study that the preservice teachers decided to curtail their students' in class questions so they could (a) avoid having to respond with additional explanations they felt unprepared to offer and (b) cover all of the content of the lesson. This study also shows that, at times, the preservice teachers avoided a particular mathematical conversation following a student query. The reasons they gave for doing so did refer to saving time and completing the lesson content but in contrast to Borko and Livingston's (1989) findings, the participants' reasons did not include a perceived lack of knowledge or a lack of ability to respond appropriately. Instead, the preservice teachers of this study indicated that it was their student judgements which led them to avoid mathematical conversations that, in their opinion, were not in the best interests of their students.

As well as knowingly withholding procedural knowledge, the participants of this study limited their enacted procedural knowledge in a second way. They either intentionally or unintentionally presented procedural explanations that would be unsuitable for problems that lay beyond the mathematical context of a particular lesson. The preservice teachers did so by choosing to teach a one-size-fits-all approach to algebraic manipulations, such as Grace's statement, "So instead of doing times and divide by first, we do plus and minus first when we're solving equations." The rules, tricks, and sequences of steps presented by the preservice teachers led their students to perform the algebraic procedures in those lessons successfully. However, the students were not exposed to the wider range of solution paths that exist for the algebraic procedures of the lessons. This finding is similar to one described by Rowland et al. (2011) who found a secondary mathematics preservice teacher attributing her students' restricted knowledge of solution paths for solving simultaneous equations to her excessive use of the rule, "If the signs are the same, then subtract; if they are different, then add" (p. 6).

Unlike the preservice teacher in the study by Rowland et al. (2011), the participants of this study did not always recognise the limitations they were placing on the procedural knowledge they enacted. Nevertheless, even when they did, they did not regret their decisions. The preservice teachers prioritised their students' success within the lesson and either did not appear to be aware of, or did not place much importance on, how the rules and tricks they were teaching their students might create difficulties when their students encountered more advanced algebra. The result of the preservice teachers' desire for success without confusion was their delivery of contextually bound mathematics content. In doing so, they failed to preserve the mathematical integrity of the content presented which is a fundamental expectation of mathematics teachers (Ball & Bass, 2009; Clemens, 1991; Harel, 2008a; Lim, 2008; Ma, 1999; Schifter, 2001; Wu, 2006). Hence, the preservice teachers in this study enacted predominantly procedural knowledge in a more restricted form than what they held themselves.

6.1.1.2 Preservice teachers can't see the forest for the trees

So preoccupied were the preservice teachers with enacting procedural steps, rules, and "tricks" (i.e., the forest) that they paid far too little attention to the concepts or ways of thinking (i.e., the trees) that underpinned the procedures. Although the preservice teachers could have regularly enhanced their procedural explanations with connections involving concepts and AWOTS, they did not take all of the opportunities available to them to enact their knowledge of AWOTS or conceptual knowledge (briefly and superficially enacted in 49% and 35% of all episodes, respectively). Descriptions of preservice teacher MCK enacted in the practicum classroom have noted a general absence of conceptual knowledge (Livingston & Borko, 1990; Rowland et al., 2011) but no previous studies have indicated the presence or absence of any mathematical ways of thinking in preservice teachers' instructional actions.

The preservice teachers' preoccupation with the "knowing how" (Mason & Spence, 1999, p. 137; Ryle, 1949/2000, p. 28) of a procedure left little room for them to draw their students' attention to "knowing why" (Mason & Spence, 1999, p. 137). Two forms of knowing why were not enacted often enough by the preservice teachers. Firstly, the preservice teachers seldom spoke of why a procedural step *can* be done, so they rarely enacted their knowledge of a mathematical concept that underpinned a procedural rule or step. The lack of connection between procedures and concepts in many episodes in this study is in direct contrast to many scholars' claims that conceptual knowledge should be explicitly taught with procedures (Eisenhart et al., 1993; Hiebert & Le Fevre, 1986; Kilpatrick et al., 2001; Schneider et al., 2011; Skemp, 1979). Secondly, the preservice teachers infrequently mentioned why a procedural step *should* be done, failing to enact their knowledge of an algebraic way of thinking that underpinned a procedural step. The absence of connections of this kind are contrary to the advice of Cuoco et al. (2010), Greeno (1978), and Harel (2008b) who recommend drawing students' attention to strategic knowledge as a procedure is learnt so that students can develop an understanding of the mathematical circumstances under which particular procedural steps would be advantageous or mathematically useful. On the whole, the preservice teachers in this study did not decide often enough to give conceptual knowledge and AWOTS explicit attention, despite implying a tacit knowledge of these MCK types in their lessons.

A tacit knowledge of certain mathematical concepts and AWOTS was noted in many of the preservice teachers' actions but they did not explicitly enact this knowledge. The preservice teachers' modelling of, and explanations for, algebraic procedures regularly implied a knowledge of (a) concepts, such as equivalence or the additive and multiplicative identities, and (b) AWOTS, such as the algebraic invariance or manipulating with purpose ways of thinking. However, for the students watching and listening to their teaching actions, those knowledges did not manifest. Explicit connections need to be made by teachers to assist students to develop those connections that are needed for a deep understanding of the subject (Hiebert & Carpenter, 1992; Kilpatrick et al., 2001; Ma, 1999; Mhlolo et al., 2012; Skemp, 1979) and that are a feature of the Australian mathematics curriculum (ACARA, 2015). Ma (1999) and Kilpatrick et al. (2001) contend that teachers need to hold their MCK in a decompressed form to be able to highlight particular aspects of lesson content to their students but this did not appear to be the case for the participants in this study. The nature of the preservice teachers' own MCK is addressed further in section 6.1.2.2.

6.1.1.3 Near enough is not good enough when preservice teachers enact MCK

The participants presented what they considered were descriptions of mathematical ideas that were "near enough" to mathematical truths. Sometimes intentionally and sometimes unintentionally, the preservice teachers presented (a) overly trimmed mathematical explanations that omitted important and necessary details (McCrory et al., 2012) and (b) verbally imprecise mathematical language. The preservice teachers deemed these versions of their MCK "near enough" to mathematical correctness and "good enough" to enact in class. However, their teaching actions reflected a lack of both specialised content knowledge and horizon knowledge, as described by Ball et al. (2008).

The restricted versions of the preservice teachers' MCK enacted in overly trimmed mathematical explanations were discussed in section 6.1.1.1. To briefly reiterate, the preservice teachers either intentionally or unintentionally restricted the MCK they enacted to suit only the examples of a particular lesson (e.g., "If $x^2 = 4$, then $x = 2.$"). By restricting their students' exposure to various solution paths or solution types to expedite their students' success with particular procedures within the lesson, the participants potentially distorted how their students might perceive the mathematical content should they attempt to apply what they had learned beyond the examples provided in the lesson.

Verbally imprecise mathematical language was revealed in the participants' lessons and compromised the quality of the mathematical content they delivered. Verbally imprecise explanations are considered poor choices by many scholars because mathematical meaning can be distorted (Dunn, 2004; Falle, 2005; Hill, Blunk et al., 2008; Rowland et al., 2011; Sleep & Eskelson, 2012; Zazkis, 2000). In this study, poor verbal language choices concerning MCK included non-mathematical informal codes (e.g., "getting rid of" a number when solving an equation), overly casual words (e.g., "timesing" "plusing" and "subbing"), and ambiguous statements that avoided more precise references to mathematical meaning (e.g., "doing things" to a pronumeral within an algebraic procedure). The possible effect of the imprecise language was that students may have found it difficult to make complete mathematical sense of many of the preservice teachers' well intentioned explanations.

This finding is similar to one of Rowland et al. (2011) who identified imprecise language choices in a secondary preservice teacher's in class explanations. The findings of

Markworth et al. (2009), concerning a secondary preservice teacher's improvement in her use of mathematical terminology over a five week practicum phase, are also consistent with the findings of this study. Together, these studies suggest that precise mathematical language is not a certainty in the preservice teacher's mathematical repertoire or even beyond the preservice teacher stage of development.

Poor language choices in mathematics are not unique to preservice teachers. Practising teachers have been found to enact casual and misleading mathematical language (Sleep & Eskelson, 2012), poorly expressed colloquial language (Heaton, 1992), or general imprecision with mathematical language (Hill, Blunk, et al., 2008; Smith, 1977). This suggests that preservice teachers may continue to enact "mathematically sloppy" (Sleep & Eskelson, 2012, p. 553) language well after graduation, unless steps are taken to break the poor verbal habits that they appear to be forming.

6.1.1.4 The silver lining: Beginnings of specialised mathematical knowledge

Specialised knowledge of algebraic procedures was occasionally demonstrated by all the preservice teachers in this study at different points of their lessons. They demonstrated an awareness of, and attention to (a) features of algebraic objects, (b) multiple solution paths, and (c) connections between procedural knowledge, conceptual knowledge, and AWOTS. The participants explicitly highlighted features of algebraic expressions or equations to their students, in accordance with recommendations by Greeno (1978) and Kilpatrick et al. (2001) that mathematical features be pointed out to students when procedures are taught. The preservice teachers' flexibility regarding different solution paths and the ease with which they were able to adapt their intended solution paths to align with student thinking during their lessons also showed a specialised knowledge of mathematical procedures needed for effective teaching (Ball et al., 2008; Star & Stylianides, 2013). Finally, the preservice teachers enacted a specialised knowledge of algebraic content when they overtly highlighted mathematical connections between procedural aspects of a solution and related mathematical concepts or pertinent ways of thinking. Connections of this kind feature regularly in the literature concerning the mathematical knowledge that teachers should teach (e.g., Baroody et al., 2007; Cuoco et al., 2010; Harel, 2008b; Hiebert & Le Fevre, 1986; Ma, 1999; Skemp, 1976).

Preservice teachers do hold some of their MCK in a decompressed form, reflecting a specialised knowledge of mathematics needed by teachers (Adler & Davis, 2006; Ball et al., 2008; McCrory et al., 2012). Although all the preservice teachers had never taught some or all of the lesson content to a class before (with the exception of Ben who taught a review lesson), they were occasionally able to identify and explicitly share with their students their knowledge of key concepts such as equivalence, AWOTS such as "manipulating with purpose," and a more flexible approach to the choice of solution path. In a study of a secondary preservice teacher by Thwaites et al. (2011), the researchers also briefly noted the preservice teacher's explicit highlighting of mathematical connections. The preservice teacher in that study pointed out particular features of quadratic equations and their connection with the associated graphical representations. Explicitly enacted connections regarding MCK significantly improved the quality of mathematical content presented by the preservice teacher in the study by Thwaites et al. (2011) and also by the participants of this study.

Although specialised content knowledge was occasionally evident in the preservice teachers' actions, there were many missed opportunities where MCK needed specifically for teaching was concerned. The preservice teachers' discussion of mathematical features, alternate solution paths, and connections involving conceptual knowledge and AWOTS were usually short lived and embedded within procedural explanations, resulting in these important ideas receiving less attention than what they deserved. The lack of conceptual connections in the practice of preservice teachers in this study was also found by Borko and Livingston (1989) in their study of two preservice mathematics teachers' instructional actions.

More explicit attention to mathematical connections would certainly have enhanced many of the preservice teachers' lessons. Nevertheless, the intermittent presence of overt connections regarding procedures, concepts, and ways of thinking in their lessons is an encouraging sign, showing the preservice teachers' desire to deliver mathematical content that might extend their students' thinking beyond Ryle's (1949/2000) notion of knowing how. Hence, the preservice teachers' enacted MCK was specialised to a degree but showed room for further development.

In summary, the participants of this study delivered mainly procedural knowledge of algebra, supported by occasional references to conceptual knowledge and AWOTS. The relative quality of the MCK enacted by the preservice teachers and its suitability for the work of teaching varied within and across the lessons. Certain aspects of the mathematical content delivered by the preservice teachers reflected a developing specialisation of MCK for teaching but other aspects of the mathematical content presented were poorly chosen and lowered the quality of the mathematics presented.

6.1.2 Major findings addressing research question 1

Research question 1 asked "What elements influence the decisions secondary preservice teachers make regarding the mathematical content knowledge (MCK) they enact when teaching lower secondary algebra?" This chapter section addresses research question 1 by drawing conclusions about decision making influences and enacted MCK. The connections discerned between particular decision making elements and enacted MCK are first synthesised in a framework demonstrating the influencing elements that lead to higher and lower quality MCK of algebra. Aspects of the framework are described, then compared and contrasted with other teacher decision making models in the literature. In the subsequent sections, three significant connections between MCK and a decision making element or a combination of elements are elaborated and discussed with reference to the literature on teacher decision making and MCK. The three connections pertain to: (a) the quality of the knowledge that preservice teachers bring to the decision making process, (b) the limited direction offered in the practicum context about the MCK that preservice teachers should deliver, and (c) the positive influence of live teacher-student interactions on the MCK that preservice teachers enact.

6.1.2.1 Framework of influences on higher and lower quality MCK of algebra

This study concluded that elements of five decision making influences, described in chapter 4, can positively or negatively impact the MCK that preservice teachers ultimately enact. Chapter 5 reported on patterns discerned between the presence of particular elements or combinations of elements and the type and quality of MCK that manifested in the ensuing episodes. Those patterns are synthesised in the diagram provided in Figure 15. The diagram indicates the potential for every influence to lead preservice teachers to enact either rich MCK (top of the diagram) or impoverished versions of MCK (bottom of the diagram) in algebra lessons.

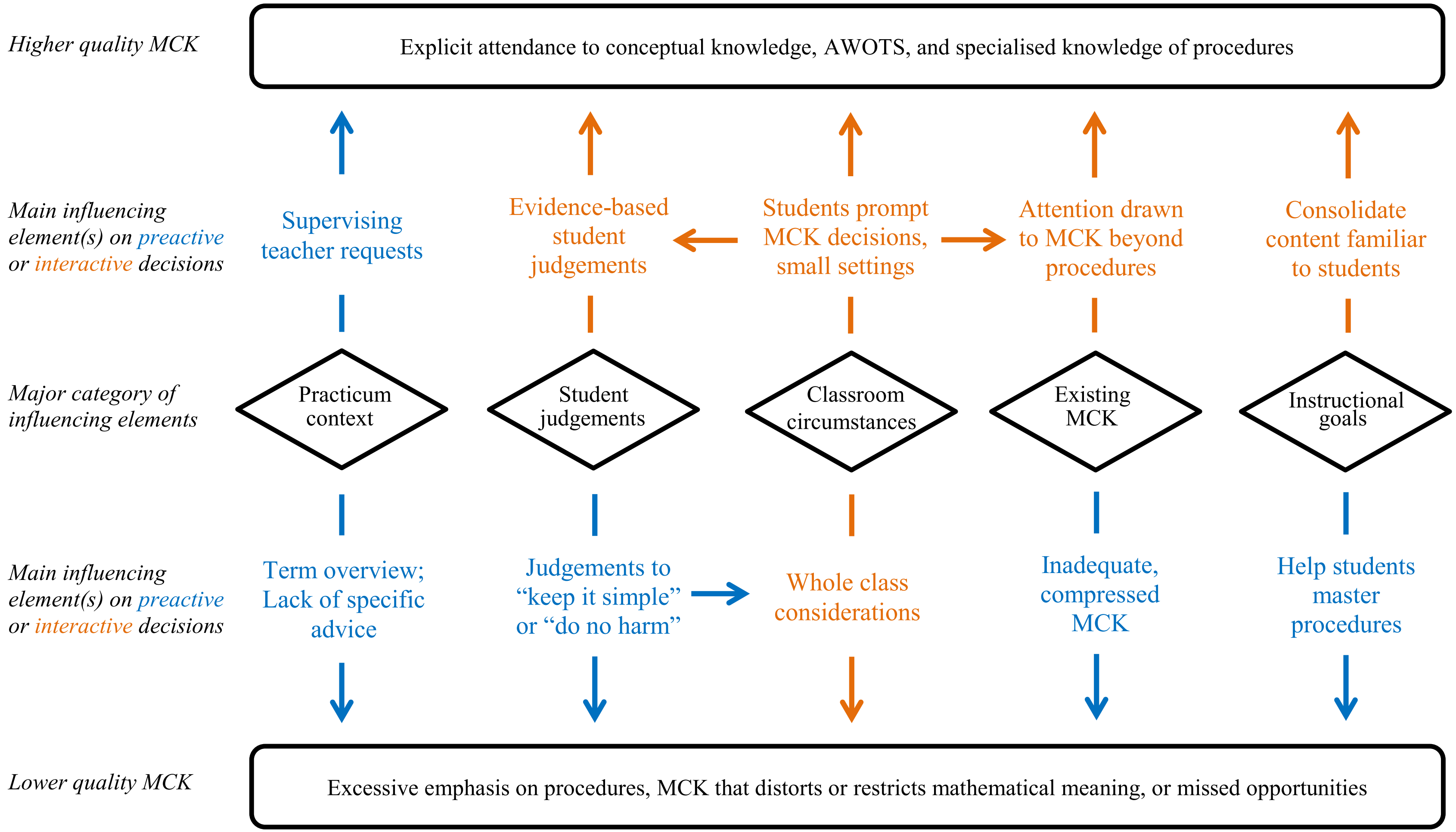


Figure 15. Framework of influences impacting the quality of MCK that preservice teachers decide to enact

The diagram in Figure 15 shows the five major categories of influencing elements in the centre row: the practicum context, the preservice teacher's judgements about students, the live classroom circumstances, the preservice teacher's own MCK, and the instructional goals formed during decision making. Above and below the centre row are particular elements belonging to each major category that tend to lead preservice teachers to enact either higher quality MCK (indicated with arrows pointing upwards) or lower quality MCK (indicated with arrows pointing downwards). Blue font is used to show those elements that relate to preservice teachers' preactive decisions in the diagram and the elements that are associated with interactive decisions in the diagram are presented in orange font. The horizontal arrows indicate the influence of one element on another in the decision making process. For example, the two orange horizontal arrows show that when student contributions prompt preservice teachers to make interactive decisions, those student contributions can also impact the preservice teachers' judgements about students and their own MCK.

As reported in chapter 4, the five major categories of influences in the centre of Figure 15 align with three constructs of Schoenfeld's (2010) model of human decision making: orienting to a situation (i.e., the practicum context and the live classroom circumstances), resources (i.e., the preservice teachers' student judgements and MCK), and goals. All other decision making models reviewed in chapter 2 (John, 2006; Leinhardt & Greeno, 1986; Shavelson & Stern, 1981; Simon, 1995; Westerman, 1991) also make reference to the influence of context and teacher knowledge on decisions and the formation of instructional goals as a component of the decision making process. Shavelson and Stern (1981) noted the presence of not only the live classroom context on teacher decision making but also the influence of institutional constraints. For the preservice teachers in this study, the constraints of the practicum context were a significant influence on their MCK related decisions. The study did not reveal additional influences on preservice teacher decision making. Rather, there were other influences identified in the literature that impact decision making that were not evident in this study.

The model of influencing elements presented in Figure 15 captures only those elements that could be observed from the lesson footage or that the participants were able to articulate. Elements that were present in the decision making models reviewed in chapter 2 but that were not evident in the data of this study were: behaviour management issues

(John, 2006; Westerman, 1991), lesson structure, including class routines (John, 2006; Leinhardt & Greeno, 1986), the preservice teachers' emotional fortitude and personality (Schoenfeld, 2010), physical resources (John, 2006; Schoenfeld, 2010), and pedagogical approaches or strategies (Shavelson & Stern, 1981; Simon, 1995; Westerman, 1991). Although these elements did not form part of the findings of this study because they were not articulated by the participants, they may well have influenced the preservice teachers' MCK related decisions to some extent.

Most notably, the preservice teachers in this study did not explicitly articulate beliefs about the nature of mathematics, algebra, or teaching when they reflected on their decisions, despite beliefs playing a significant role in teacher decision making (Shavelson & Stern, 1981; Schoenfeld, 2010; Simon, 1995; Westerman, 1991). Westerman (1991) found in a study of primary mathematics preservice teachers that their beliefs were a far stronger influence on their teaching decisions than their content knowledge but this was not the case for the participants of this study who regularly drew on their content knowledge to inform their decisions. Nevertheless, the findings of this study may indicate that while most aspects of MCK related decisions are not visible to an observer, some aspects of MCK related decisions may not be evident to the preservice teachers themselves. Sullivan (2003), for example, when reflecting on his own teacher education experience noted that he had never reflected on the beliefs he held regarding the nature of mathematics during his tertiary studies.

For the participants in this study, their beliefs and the influence of those beliefs on certain aspects of their teaching practice may be held in a tacit form. They were not specifically articulated in their interviews but did permeate through the other influence categories when the participants referred to their mathematical preferences or when they offered their interpretations of live events or the practicum context. Schoenfeld (2010) explains that a person's orientations including beliefs, values, and preferences interact with other elements in the decision making process and this appears to be the case for the participants in this study. Therefore, the absence of the preservice teachers' beliefs from the findings of this study may be a reflection of the tacit form in which preservice teachers hold their beliefs, rather than an indication that beliefs do not form part of the MCK related decision making process.

The elements in the decision making process that were found to lead to higher and lower MCK in this study could not be compared with those of other decision making models. This was because no decision making models make reference to particular elements that lead to stronger or weaker instructional practice, let alone the MCK manifesting in teaching actions. Where possible, however, comparisons are made with the findings of studies investigating secondary preservice teachers' MCK in their practicum teaching practice.

When preservice teachers deliver mathematical content that is closer to the "ideal," as espoused in the literature on teaching secondary algebra (e.g., McCrory et al., 2012; Yakes & Star, 2011), certain influences tend to be present. For preservice teachers to enact high quality MCK, they must at least already possess the knowledge themselves or may, on the advice of a supervising teacher, learn additional MCK for their lesson. Further, they need to hold that knowledge in a decompressed state so that they are aware of its existence. When concepts and AWOTS are on the mathematical radar of the preservice teachers, they are more able to establish goals that focus on mathematical content beyond simple mastery of procedures. Their attending to connections involving conceptual knowledge, AWOTS, or a flexible approach to procedures results in the delivery of higher quality mathematical content. The enactment of more desirable MCK can also be influenced by the preservice teachers' students when they prompt discussions of mathematical procedures regarding particular points of confusion. When preservice teachers intend to address those points of confusion, they refine their knowledge of students' mathematical needs and better versions of their MCK emerge in their explanations. Opportunities for these types of situations tend to occur when preservice teachers are intending to consolidate their students' understanding of mathematical content that is familiar to students, in small instructional settings.

Poor MCK, too, was associated with particular decision making elements. Where inadequacies exist in the preservice teachers' own MCK, they are reflected in impoverished forms of content delivery. Where questionable judgements concerning the preservice teachers' views of students influence the preservice teachers' decisions, the MCK enacted tends to be less precise, more contextually bound, and lacking in conceptual knowledge and AWOTS. This situation occurs more often when the preservice teachers are teaching in a whole class instructional setting. As the preservice teachers attempt to meet the mathematical needs of large numbers of students at once, they rely on the MCK and student judgements they hold to inform their decisions about the content whole class cohorts are capable of understanding. Those decisions often involve the establishment of student focused goals such as "To avoid student confusion" and lead to more superficial versions of MCK.

Intentional and unintentional omissions of mathematical content lead to poorer versions of MCK being enacted. Content focused goals that focus on students mastering algebraic procedures often lie behind the preservice teachers' enactment of MCK that is unintentionally devoid of crucial knowledge concerning concepts and AWOTS. The preservice teachers' preoccupation with procedures appears to be exacerbated by macro lesson goals that prioritise procedural competence, developed from the school term overviews. Preservice teachers' decisions to intentionally withhold a wider range of solution paths for certain algebraic procedures tends to be influenced by the practicum context and the preservice teachers' judgements about students. Those aspects of MCK perceived as unnecessary, time consuming, or problematic are intentionally omitted. The MCK that does manifest is more automated and less transferable to different mathematical contexts as a result.

Looking across the influences found in this study that lead to MCK enactment, three major conclusions can be drawn. First, the quality of the mathematical content that preservice teachers deliver in a live classroom is strongly influenced by the quality of the preservice teachers' own resources (i.e., their MCK and judgements about students) that they themselves bring to the decision making process. Second, elements of the practicum context have the potential to influence preservice teachers' MCK related decisions but only if strong direction exists in that context from the supervising teacher or school documents, such as the term overview. Third, student contributions are one of the most valuable resources available to preservice teachers where MCK is concerned. As students react to preservice teachers' delivery of mathematical content, their responses, in turn, encourage the preservice teachers to rethink their perceptions of students' mathematical needs, rethink the MCK they are enacting, and improve the quality of the mathematical content they present. Each conclusion is elaborated below.

6.1.2.2 The quality of the preservice teachers' resources directly impacts the quality of the MCK they enact

When preservice teachers make MCK related decisions, those decisions are informed by elements within and outside their control. Preservice teachers are directly responsible for the adequacy of their mental resources that they bring to the decision making process. In this study, the most influential mental resources that impacted preservice teachers' MCK related decisions were their own MCK and the judgements they held about mathematics students. This section presents the conclusions drawn from this study about the quality of both resources and their impact, individually and collectively, on the quality of the preservice teachers' subsequent decisions.

The findings of this study indicate that preservice teachers do not regularly take further steps to strengthen their MCK of algebra when preparing for their lessons. In general, they believed there was no need. The only times that the participants learnt new mathematics content prior to their lessons was when they were directed to do so either by their supervising teacher or as a result of reviewing unfamiliar textbook procedures that were to form part of their lessons. The minimal development of the participants' MCK in the planning stages of their lessons suggests that the MCK that preservice teachers hold outside of the practicum context may closely resemble the MCK they draw upon within the practicum context. Studies investigating preservice teacher MCK outside the classroom context (e.g., Ball, 1990; Bryan, 1999; Even, 1993; Goos, 2013; Tatto et al., 2012) can, to a degree, offer an indication of the MCK preservice teachers hold as they enter the practicum context and what they might possibly choose to enact. However, decontextualised descriptions of preservice teacher MCK cannot identify the MCK that preservice teachers emphasise, reject, restrict, or develop while teaching a mathematics lesson. This study found that the MCK enacted by the participants in the classroom context was similar to, but not identical to, the MCK they enacted outside the classroom context (i.e., during their interviews).

The MCK enacted by the participants during their lessons shared similarities and differences with the MCK they enacted in their interviews. The preservice teachers sometimes did not enact all the MCK they possessed concerning a particular topic, concept, or skill; rather, they enacted a version that they decided suited the practicum context, their students' mathematical needs, and live classroom circumstances. Hence, the

MCK of algebra that preservice teacher enact during a lesson is similar to, but not the same as, the preservice teachers' own MCK of algebra.

The preservice teachers used imprecise mathematical language in their lessons and interviews, reflecting limitations in their own MCK. The preservice teachers' lack of attention to precise verbal explanations appears to stem from their underdeveloped MCK. The preservice teachers appear to hold their knowledge of algebraic symbols and procedures expressed in written form far more precisely than they do their verbal equivalents, reflected in their imprecise explanations during their lessons and interviews. Markworth et al. (2009), who found that a preservice teacher's use of mathematical language improved as a result of her practicum experience, contend that precise mathematical language is not necessary for the non-teacher and is a facet of specialised content knowledge that can be developed over time in the practicum context. In this study, only a snapshot of preservice teachers' use of language was taken but the findings indicate that further development of preservice teachers' verbal capacities is needed.

The participants in this study privileged procedural knowledge in their classroom actions and in their interviews. It appears that the preservice teachers' own MCK may lack considerable depth and hinder their efforts to deliver high quality mathematical content. The analysis of the episode data (interview and classroom) suggests that superficial knowledge of procedures appears likely to be the main reason for multiple opportunities to enact related conceptual knowledge or AWOTS being squandered. It was not always possible, however, to identify which aspects of a preservice teachers' MCK were either missing altogether or so compressed that they were unable to speak explicitly about them. A notable absence of conceptual knowledge and AWOTS was nevertheless the result. The interview reflections did suggest, however, that both possibilities were likely for different preservice teachers at different times in their lessons.

Interview reflections where preservice teachers encountered difficulties describing mathematical concepts adequately suggest that preservice teachers' MCK of algebra may lack conceptual depth. Insufficient conceptual knowledge has been a finding commonly reported in decontextualized measures of preservice teacher MCK (e.g., Bryan, 1999; Even, 1993; Plummer & Peterson, 2009; Thomas, 2003). It is therefore possible that certain aspects of the preservice teachers' conceptual knowledge were absent and their absence removed even the possibility that they could decide to enact those aspects of conceptual knowledge in their lessons.

The preservice teachers' reflections of other episodes indicated a compressed, and therefore barely accessible, knowledge of certain concepts and ways of thinking. The preservice teachers struggled to explain student problems related to concepts or ways of thinking, including the additive inverse property and the algebraic invariance way of thinking, despite performing procedures in ways that showed a tacit knowledge of the same mathematical ideas during their lessons. It seems likely that preservice teachers hold compressed forms of certain aspects of their MCK. Compressed mathematical knowledge is described by certain scholars as favourable for mathematics students to develop (Gray & Tall, 1994; Harel & Kaput, 1991; Kilpatrick et al., 2001; Ma, 1999; Sfard, 1991) but is described by others as problematic for the work of teaching (Adler & Davis, 2006; Ball & Bass, 2000; Ball et al., 2001).

Holding compressed knowledge makes it more difficult for a skilled performer to appreciate elements of a novice performance (Cohen, 2011). It would appear that the notion of compressed knowledge and the difficulties experts encounter when attempting to make sense of novices' actions may be applicable to the performance of the participants in this study. While they were pre-novices in teaching, they were experts in lower secondary algebra. For example, they tended to view the two arithmetic operations of addition and subtraction in the algebraic context only in terms of addition. Subtraction was regularly treated as addition of a number's additive inverse in their teaching actions and in their interview responses. Gray and Tall's (1994) description of a *procept* encompasses both meanings of the minus symbol (i.e., subtraction and negativity) and a flexibility to move effortlessly between the two meanings when working mathematically. Thinking *proceptually* is necessary for success in mathematics, according to Gray and Tall (1994) and it is likely that the preservice teachers would have developed this mathematical way of thinking. So compressed was their knowledge of the additive inverse that the participants were often unable to articulate in the interview why their students were having difficulties following their explanations that treated subtraction symbols as a sign of negativity. The compressed nature of some aspects of the preservice teachers' MCK made it difficult for them, at times, to respond adequately to some of their students' difficulties.

Compressed MCK also may have contributed to the preservice teachers' inattention to conceptual knowledge and AWOTS in their lessons and interviews. Berliner (2001) contends that experts in a field "are not consciously choosing what to attend to and what to do" (p. 24). For the preservice teachers in this study, while they were aware of certain elements involved in performing a procedure (i.e., rules and steps), they did not appear to be consciously aware of other elements (i.e., algebraic ways of working or mathematical concepts) that were implied in the procedures they modelled. Therefore, as Figure 15 shows, poorer quality MCK in the preservice teachers' lessons, including an absence of conceptual knowledge and AWOTS, could be attributed to inadequate MCK but could also be attributed to the state of expert knowledge they hold regarding their knowledge of lower secondary algebraic procedures.

A major finding of this study is that preservice teachers' MCK of algebra may be decompressed as a result of teaching an algebra lesson. Student questions, comments, and written work appeared to lead to a shift in the way the preservice teachers held their MCK. Although the participants' teaching actions showed evidence of a tacit knowledge of certain concepts and AWOTS, they tended to explicitly enact those types of MCK when student prompted events in the classroom drew their attention to them. The preservice teachers even expressed surprise at having to expressly point out AWOTS that they hadn't planned to teach. Their reflections suggest that the purpose for, and concepts underpinning, procedural steps may be so tightly packed together with the steps themselves that they appear one and the same to the preservice teachers. Student comments seem to highlight an element of mathematics that is, up until that moment, hidden from the preservice teachers and allow the preservice teachers to decompress their MCK, becoming more aware of MCK elements that they instinctively or implicitly know.

The phenomenon of MCK decompression during a lesson extends the descriptions of knowledge development that are provided in the teacher decision making frameworks reviewed in this study. The decision making frameworks of Schoenfeld (2010), Shavelson and Stern (1981), Leinhardt and Greeno (1986), and Simon (1995) make references to knowledge developing as a result of live classroom circumstances but the descriptions of knowledge development focus on the teachers' developing knowledge of their students' understandings and not their own subject matter knowledge. Although the findings of this study revealed that judgements about students can develop during a lesson, the preservice teachers' own MCK was also developed as a result of interactions with students within a lesson.

Live student interactions led the preservice teachers to re-examine the MCK they enacted in the same way that Berliner (2001) posits that experts reflect on their own performance only when something atypical occurs. The unplanned responses required of the preservice teachers as they reacted to student prompted events were likely the result of unpacking and reorganising their compressed knowledge of mathematics. Ball et al. (2008), Lim (2008), and Ma (1999) contend that unpacking mathematical ideas is essential for teachers so that features of mathematics can be made apparent to students. The practicum classroom context, including mathematical conversations with students, appears to encourage preservice teachers to unlock crucial elements of compressed mathematics knowledge. Teaching an algebra lesson can allow preservice teachers opportunities to reconfigure their own MCK into more unpacked and, therefore, more suitable forms for teaching.

One of the differences between the MCK enacted by the participants in their lessons and in their interviews suggests that secondary mathematics preservice teachers lack adequate horizon knowledge. Restricted forms of procedural knowledge that were present in the preservice teachers' lessons but not in their interviews were described earlier in this chapter. The restricted version of MCK that the preservice teachers chose to teach to their students may be, in part, the result of the preservice teachers lacking the horizon knowledge needed to "see" how the mathematical content of a particular lesson will extend to future mathematical ideas (Ball & Bass, 2009). A lack of horizon knowledge seems likely because the preservice teachers appeared unable to see beyond the lesson being taught and place lesson content within a broader mathematical context of three or four weeks.

Underdeveloped horizon knowledge may be associated with the short-term nature of preservice teacher planning, which Borko and Livingston (1989) found in their study. One preservice teacher in their study commented that she didn't "plan much further than tomorrow" (p. 486). Although the participants of this study did work from term overviews, unlike the participants in Borko and Livingston's (1989) study who planned ahead by looking at the next few pages of the textbook, the short-term nature of the practicum does not appear to help preservice teachers develop horizon knowledge. This facet of MCK appears to be underdeveloped for preservice teachers and not enhanced by short-term practicum placements.

Preservice teachers' own MCK impacts their MCK related teaching actions but a second mental resource, the judgements they hold about mathematics students, is also a significant influence. The participants of this study drew upon the judgements they held about how students learn mathematics and students' mathematical needs to inform their choice of goals and choice of MCK to enact. The judgements held by the preservice teachers manifested in their desire to value and encourage their students' contributions but also to teach the content that they wanted their students to learn. Cobb, Yackel, and Wood (1991) highlight the tensions that teachers experience between "encouraging students to build on their informal mathematical ways of knowing and attempting to teach them the institutionally sanctioned formal [mathematics]" (pp. 84-85). The preservice teachers' establishment of micro goals reflected this tension as they formed content focused and student focused micro goals at different points of their lessons.

The quality of the participants' judgements about students in this study was variable, impacting their MCK related decisions in positive and negative ways. Many of the judgements articulated by the preservice teachers in their interviews were questionable in nature and, at times, led to poor decisions regarding MCK enactment. The preservice teachers' judgements that students should never be confused (i.e., "do no harm") and should only be exposed to simple, uncomplicated mathematical ideas (i.e., "keep it simple") tended to lead to poorer quality mathematical content being presented (see Figure 15). Those same judgements tended to have a stronger influence on preservice teachers' interactive decisions in larger instructional settings than on their interactive decisions in smaller settings, indicated in Figure 15 by the blue horizontal arrow. The combination of poor student judgements and a lack of horizon knowledge often underpinned the preservice teachers' intentional and unintentional restriction of enacted MCK which was limited in its application to mathematical problems or topics that lay beyond the immediate lesson context.

In addition to the questionable student judgements identified in the data, more robust student judgements, based on information gleaned by the participants during their lessons, were associated with interactive decisions to enact higher quality MCK. This finding aligns with Simon's (1995) description of teacher decision making as improving over the course of a lesson. Simon theorises that hypothetical judgements about students are continually developed during a lesson as the teacher interacts with students and those refined judgements should lead to improved instructional decisions. The positive influence of classroom interactions on the development of the preservice teachers' student judgements and, in turn, on their MCK related decisions and actions are elaborated in section 6.1.2.4.

These two significant mental resources - MCK and student judgements – and elements of the practicum context often competed for the preservice teachers' attention. They regularly weighed up the pros and cons of teaching different aspects of mathematical content to their students. When they did so, the preservice teachers' own mathematical preferences, an aspect of their MCK, were rarely privileged over mathematical content that the preservice teachers judged was more suited to the kind of content to which students need exposure. The privileging of student judgements over the preservice teachers' MCK did not always result in high quality MCK being enacted, however, because the judgements about students, as described earlier, were not high quality judgements.

The results of this study suggest that because preservice teachers lack teaching experience, they rely on flawed assumptions about students that are rooted in their own mathematical learning experiences. They may in fact be better off relying on their content knowledge to guide their choice of MCK when planning their lessons, rather than the hypothetical judgements they hold about students. This is particularly so if they are directed to focus on conceptual underpinnings of procedures or ways of thinking. Relying more heavily on content knowledge to influence instructional decisions is not a viable long term solution for teachers, as pedagogical content knowledge (PCK) has been more closely associated with effective teaching (Baumert et al., 2010; Goos, 2013). In the interim, however, as preservice teachers build up a more robust understanding of how students learn mathematics, they may find more success in practicum lessons by paying careful attention to the mathematical content they plan to deliver.

6.1.2.3 Practicum offers limited direction regarding the MCK preservice teachers should enact

The practicum context was a category of influencing elements that were beyond the direct control of the preservice teachers but which they were expected to, and did, consider when making MCK related decisions. Shavelson and Stern (1981) and John (2006) identified the school or institutional context as an influence on practising and preservice teachers' decisions and in this study, the practicum context did influence the MCK enacted to a degree. Sensitive to the practicum context in which they were teaching, the preservice teachers were eager to meet practicum requirements. They did so by delivering content that (a) suited the mathematical ability of their class, (b) aligned exactly with the content of the term overview and mostly with the class textbook, and (c) satisfied all their supervising teachers' expectations.

Studies of secondary mathematics preservice teachers within the practicum context have identified two practicum elements found in this study and one additional element that impact instructional decisions. Advice and feedback from supervising teachers (Borko & Livingston, 1989; Livingston & Borko, 1990; Markworth et al., 2009) and the class textbook content (Borko & Livingston, 1989; Markworth et al., 2009) have been reported as influences on the mathematical content that secondary preservice teachers decide to deliver. The studies by Rowland et al. (2011) and Livingston and Borko (1990) also identified school assessment items as an influence because the preservice teachers were required to discuss their students' attempts on past exams or assignments during their lessons. This influencing element was not evident in the data of this study because no lessons involved students completing an assessment item or reviewing their attempts at one. However, two participants, Ben and Thomas, did reflect that their class had an exam coming up and they were aware of the types of questions that might be on the exam when they chose the content to deliver in their lessons.

The practicum context was a strong but general influence on the MCK that preservice teachers decided to enact. The influence of the practicum elements manifested in the highly procedural macro goals of the lesson (lesson objectives), formed by the preservice teachers from brief content statements provided in the school term overview, and the subsequent formation of multiple episode goals to develop students' knowledge of algebraic procedures. Goals that emphasised mastery of procedures, often formed in preactive decisions, tended to lead to poorer quality MCK in the preservice teachers' lessons (see Figure 15).

Where specific advice was given by supervising teachers regarding particular facets of content delivery, the preservice teachers willingly complied. However, in this study specific MCK was rarely enacted and more often omitted on the advice of the supervising teachers. Only one preservice teacher chose to learn additional MCK before a lesson in response to a supervising teacher's request but only one supervising teacher made a request of this kind. Given the weight that supervising teachers' advice carried with the preservice teachers, it is likely that more specific requests regarding mathematical content would have led to more preservice teachers focusing on the mathematical ideas highlighted by the supervising teachers (reflected in Figure 15). Markworth et al. (2009) found in their study of a secondary preservice teacher and her supervising teacher that the supervising teacher had not expected the preservice teacher to develop her MCK during the practicum even though this did occur. Markworth et al. (2009) concluded that supervising teachers may assume that preservice teachers' knowledge is sufficient and may not offer specific advice about mathematical content. The opinions of the supervising teachers were not sought in this study but the lack of specific advice concerning content delivery is consistent with the findings of Markworth et al. (2009).

The textbook content provided the most specific information regarding MCK in the practicum context but textbook content was a practicum element that preservice teachers treated as relatively discretionary; nevertheless the MCK they decided to enact was often similar but not always a replica of the textbook content. Hence, the textbook was a relatively strong influence but it did not specifically influence the preservice teachers' decisions to enact the better aspects of MCK that they possessed. Borko and Livingston (1989) concluded that preservice teachers struggle to make priority decisions regarding textbook content when planning their lessons but for the participants in this study, they were left to make those decisions on their own. Although a textbook may include significant references to mathematical concepts or ways of thinking, if preservice teachers are not able to make high quality choices from textbook content, as Borko and Livingston (1989) concluded, references to mathematical ideas beyond mastery of procedures may go unnoticed.

Overall, the preservice teachers received little guidance about the type of MCK that should feature in their lessons from the practicum context beyond the algebra topic they were required to teach. Elements of the practicum context rarely offered the preservice teachers specific mathematical advice regarding key ideas to which their students should be exposed, reflecting the "benign" practicum experience encountered by many preservice teachers, according to Lewthwaite and Wiebe (2012, p. 49). A lack of mathematical direction was evident in the very brief content descriptions provided to the participants in the school term overview. The freedom offered by the supervising teachers to the participants when they prepared their lessons was also problematic because a lack of specific mathematical direction accompanied the free reign given to the preservice teachers. The only specific ideas given to the participants to support their lesson preparation lay within the textbook content but the participants were not directed to any particular ideas in the textbooks and tended to focus only on the procedures that they wanted their students to master. Consequently, the preservice teachers relied primarily on their underdeveloped judgements about students and their own MCK to make MCK related decisions, with the assistance of their students when live interactions took place.

6.1.2.4 Students bring out the best of preservice teachers' MCK

Student prompted classroom events had a very positive impact on the quality of MCK that the preservice teachers enacted. The preservice teachers always enacted MCK with the best of intentions where students and their needs were concerned but it was the students themselves who identified their own needs most successfully. As they planned their lessons, the preservice teachers made hypothetical assumptions about what their students' mathematical needs would be and how their students might learn mathematics most effectively. As the lessons unfolded and unplanned episodes took place, students gave an indication of their real mathematical needs, via questions, comments, and written work and the preservice teachers enacted their strongest aspects of MCK in response.

The findings of this study regarding high quality MCK in unplanned episodes contrasts with previous studies of preservice teachers' interactive decisions. In the studies by Borko and Livingston (1989; Livingston & Borko, 1990) and Westerman (1991), the preservice teachers struggled to respond adequately to live student requests when teaching practicum lessons. The preservice teachers also indicated that they lacked confidence regarding their content knowledge when they were called on by students to create spontaneous mathematical explanations. The difference between the findings of Borko and Livingston's studies and the findings of this study may be explained by the mathematical content being presented in the lessons. The preservice teachers in Borko and Livingston's studies taught lessons on analytic geometry, measurement (volume of solids), and calculus whereas the preservice teachers in this study taught relatively simple algebraic procedures, lesson content that they felt very confident presenting. In Westerman's (1991) study, the preservice teachers were primary preservice teachers so they may not have had the level of content knowledge of the secondary mathematics preservice teachers in this study.

The strong and positive impact of student prompted events on enacted MCK in this study appears to stem from the development of the quality of preservice teachers' student judgements and their MCK when they respond to real students. In Figure 15, this phenomenon is captured by the two orange horizontal arrows in the diagram. When interactive or unplanned decisions were made by the preservice teachers they relied upon student judgements to inform their decisions but they had the added benefit of live student responses to hone their judgements about what content was needed by their students. They became more aware that students need to be explicitly exposed to connections involving concepts, ways of thinking, and particular features of procedures and subsequently taught those connections. The students' mathematical offerings also appeared to help the preservice teachers unpack their compressed forms of algebraic knowledge because the participants were prompted to reflect on the mathematics they had taught. As they did so, they became more cognisant of key features of the content they were delivering and were able to point out those features to their students, enacting higher quality MCK. The feedback loop formed by students responding to the preservice teacher's teaching is identified in Simon's (1995) *Mathematics teaching cycle* model.

Simon's (1995) teacher decision making model is presented as a cycle where teacher knowledge informs teaching actions and interactions stemming from those teaching actions develop teacher knowledge. The development of the preservice teachers' resources in this study aligns with Simon's description of teacher knowledge that develops during instruction. Although interactions with students may not strengthen preservice teacher knowledge in lessons pertaining to all mathematical topics (e.g., Borko & Livingston, 1989; Livingston & Borko, 1990), they do appear to benefit preservice teachers when they teach algebra. This finding indicates that preservice teachers benefit greatly from teaching algebra during practicums.

In this study, high quality MCK enacted in response to student prompted events was more prominent in small instructional settings and in lesson phases where the preservice teachers were looking to consolidate content (see Figure 15). When preservice teachers are revisiting lesson content in relatively private conversations, they are perhaps more receptive to student ideas and more prepared to negotiate mathematical meaning with them. The preservice teachers' enactment of higher quality MCK in smaller instructional settings and lower quality MCK in larger settings (see Figure 15) may be explained by Benner's (1982) claim that novices find it difficult to take in all the information available to them in a particular context. Jackson (1968) explains that for students, "learning to live in a classroom involves…learning to live in a crowd" (p. 10) and it seems that the same could be said for pre-novice teachers who are relatively inexperienced in mediating content with so many students at once. It was not possible to conclude from the data in this study whether preservice teacher discomfort, a lack of stronger student judgements, or a desire to remain in their mathematical comfort zone led to weaker MCK in large instructional settings. What can be concluded is that within smaller settings, during student interactions, the best of preservice teacher MCK tended to manifest.

The conclusions drawn from this study show the advantages of studying preservice teacher MCK "in situ." Studies of preservice teacher MCK located outside the classroom context (Ball, 1990; Bryan, 1999; Even, 1993; Goos, 2013; Tatto et al., 2012) are unable to capture the dynamic nature of how preservice teachers know mathematics when they are teaching. Lave and Wenger's (1991) portrayal of mathematical knowing as "activity by specific people in specific circumstances" (p. 52) aptly describes the participants' delivery of particular aspects of MCK to different students in response to different classroom events. This kind of mathematical knowing can only be investigated in studies located in the classroom. The findings of this study support Borko and Livingston's (1999) claim that teaching performance is, in part, improvisational in nature and mathematical knowledge can be called upon in the moment as teachers respond to classroom events. Hence, measures of MCK that are ascertained in the classroom context offer a valuable perspective of preservice teacher MCK that studies located outside the classroom cannot provide.

6.2 Theoretical explanations for findings

This study sought explanations for why preservice teachers enact particular MCK by investigating the decisions that lead to MCK related teaching actions. Drawing on the literature in chapters 2 and 3, this chapter section seeks theoretical explanations for the decisions themselves, to better understand why preservice teachers make particular MCK related decisions. The major findings of this study can be explained by (a) the situated nature of preservice teacher MCK, (b) preservice teachers' experiences with lower secondary algebra prior to practicum teaching, and (c) preservice teachers' limited opportunities to develop evidence-based judgements about mathematics students.

6.2.1 The situated nature of preservice teachers' enacted MCK

Lave and Wenger (1991) posit that "understanding and experience are in constant interaction – indeed are mutually constitutive" (pp. 51-52). The findings of this study support Lave and Wenger's (1991) claim because they indicate that preservice teacher mathematical understandings and the experience of live teaching shape each other. For the preservice teachers in this study, the MCK they enacted was situated within a live classroom context, nested within a broader practicum context. This study found that the MCK enacted by the preservice teachers led to particular mathematical discussions taking place in the lesson. Hence, the MCK delivered by preservice teachers contributed to the context of the lesson. However, the reverse scenario was also true. Many of the preservice teachers' MCK related actions could be traced back to decisions that were influenced by elements of the practicum and classroom contexts (described in section 6.1.2.) showing how context can lead to the emergence of new ideas (McNeil & Alibali, 2005). This section elaborates these ideas, showing how the practicum and classroom context contributes to MCK related decisions just as their decisions lead to actions that shape the classroom context.

When preservice teachers enter the practicum setting as pre-novices, they engage in "legitimate peripheral participation" (Lave & Wenger, 1991, p. 34) within the school community. Lave and Wenger (1991) describe legitimate peripheral participation as a type of engagement within a social practice (in this study, the practice of teaching) where learning is an integral component of that engagement. That means that preservice teachers, within the practicum setting, engage in the practices of mathematics teaching as mathematics teacher apprentices and in doing so, they learn how to be mathematics teachers.

When preservice teachers enter a school community, they develop an idea of how people who are part of that community behave (Lave & Wenger, 1991) and they begin to align their practices with those in the community (Cavanagh & Prescott, 2007). Preservice teachers learn that the mathematical content they deliver must conform to institutional constraints (Shavelson & Stern, 1981), including an adherence to the content presented in term overviews.

In this study, so strong was the preservice teachers' desire to get through all of their lesson content and progress through the term overview topics, that the practicum setting appeared to contribute to the weaker forms of MCK (i.e., automated, restricted, and superficial) that they taught. Skemp (1979) contends that instrumental learning that focuses only on procedural competence is quicker to learn so this may partially explain why the preservice teachers paid so little attention to developing their students' relational understanding of procedures (Skemp, 1979). The preservice teachers' desire to keep up with the work to be covered in the term overviews is echoed by Westerman's (1991) framework of decision making for preservice teachers which identifies curriculum guidelines as a major influence. Therefore, preservice teachers' membership within the school community of practice informs their decisions regarding the mathematical content that they deem is most suitable to present in schooling situations.

The resources that preservice teachers bring to the practicum setting influence their participation within the school community. A teacher's knowledge and actions within the community of practice of mathematics teaching contributes to the community, just as the membership within the community develops their knowledge and shapes their teaching actions (Cavanagh & Prescott, 2007). Influencing elements of the practicum context were influenced themselves in this study by the preservice teachers' resources, namely, their own MCK and their judgements about students. The preservice teachers drew on their resources when choosing which aspects of the practicum elements to emphasize in their lessons. For example, the participants were prepared to ignore parts of textbook content that failed to align with their own MCK preferences or their judgements about content which their students should be taught.

The preservice teachers' resources also contributed to the interactions that took place within the classroom community during their lessons. The MCK that the preservice teachers chose to share with their students, informed by elements of the practicum context but also their own resources, formed the basis of classroom interactions. The plethora of procedural knowledge enacted by the preservice teachers set the tone for classroom discussions that concentrated on the procedures they chose to emphasise. Hence, the resources that preservice teachers hold when they enter the school community and that inform their decisions as they participate within that community impact the community of practice of which they are a part, as Cavanagh and Prescott (2007) claim. This study concluded that the quality of the resources that preservice teachers hold, and that shape the classroom community of practice, are not fully developed for teaching and the possible reasons for the underdevelopment of those resources are offered in sections 6.2.2 and 6.2.3.

Preservice teachers' participation within the school "community of practice" (Lave & Wenger, 1991, p. 98) increases over time as they learn more about the practice of mathematics teaching (Cavanagh & Prescott, 2007). Consequently, the preservice teachers' own resources that they bring into the practicum setting develop with time and experience in the school community. This study found that by planning and teaching a single lesson, the preservice teachers' participation within the practicum community of practice led to a development in their MCK and their judgements about students.

When preservice teachers engage in planning a practicum lesson, opportunities for the development of their own resources may present themselves. In this study, elements of the practicum context occasionally guided the preservice teachers to learn additional MCK to enact. One preservice teacher's (Sam's) compliance with his supervising teacher's requests and his review of textbook content led him to learn new mathematical content. This study also showed, however, that if preservice teachers are not directed to develop their MCK for their lessons they may not be aware that their MCK is less than adequate for the lesson ahead.

While engaging in the live act of teaching mathematics, preservice teachers can also develop their resources. Simon's (1995) cyclic framework of teacher decision making shows that teachers apply the knowledge they have of mathematics and their students' understandings when the lesson begins. Teachers can then develop their knowledge as a consequence of participating in a live lesson and improved instructional decisions should follow as the lesson progresses. This cycle is consistent with the findings of this study. The participants' preactive decisions led them to emphasise algebraic procedures for most of their lessons and prompt their students to talk about mathematical procedures, showing how the participants' resources can shape the classroom community. Although students responded to the participants' teaching actions with procedures in mind, their verbal and written contributions that manifested as a result of the preservice teachers enacting procedural knowledge led to additional knowledge concerning ways of thinking or mathematical concepts being enacted by the participants.

This phenomenon may be explained by the change in the mathematical direction of the lesson that occurred when the preservice teachers responded to student prompted events. As students became the unintended directors of the mathematical ideas mediated in certain episodes, their questions and comments prompted the preservice teachers to consider the lesson content from a new perspective, helping unpack previously hidden concepts and ways of thinking. The preservice teachers' judgements about what students needed were also refined to accommodate their students' live contributions. By engaging in the mathematics classroom community of practice, the preservice teachers were able to develop their mental resources to better suit the work of teaching.

The change in the preservice teachers' MCK and student judgements shows how mathematics knowledge that teachers use in class is situated and attuned to the needs of their instructional practice (Adler, 2005; Ball & Bass, 2000). When preservice teachers are made aware that the resources they hold are flawed, they are willing to develop those resources. Preservice teachers and their students do not always recognise when the preservice teachers' resources are inadequate, however, as the participants' persistent use of imprecise mathematical language in this study shows. For the preservice teacher, who holds limited resources and lacks experience in teaching situations, the live teaching experience provides valuable opportunities for them to develop better resources for teaching. Specific advice from an experienced supervising teacher would also assist preservice teachers to develop stronger MCK and PCK.

Overall, preservice teacher knowledge in the classroom should not be considered from a static position but instead from an evolving, contextual one (Mason & Spence, 1999) as knowledge is continually constructed, reordered, and reorganised in the mind of an individual (Bodner, 1986; von Glasersfeld, 1984). Engaging in the social practice of the practicum can positively and negatively influence preservice teachers' decisions about enacting MCK.

6.2.2 Preservice teachers' learning experiences with lower secondary algebra

The previous section explained that the resources brought by pre-novice teachers to the practicum setting will influence the quality of instructional decisions that they are able to make. Those resources are developed outside the practicum setting and must be applied by the inexperienced teachers in their teaching practice. The secondary mathematics preservice teachers in this study had many years' experience in working with lower secondary algebra topics before they shared that knowledge with their students in their practicum lessons. The preservice teachers' overemphasis of algebraic procedures during their lessons and their lack of regard for, or understanding of, the concepts and ways of thinking that underpin those procedures may have root in the resources they had developed as a result of their own experiences learning secondary and tertiary mathematics.

When preservice teachers teach algebra, MCK is firstly, a critical resource drawn upon to inform MCK related decisions and secondly, it is the product of those decisions, manifesting in classroom actions. The MCK enacted by the preservice teachers in this study suggests that their MCK is somewhat underdeveloped in certain areas and they do not yet possess what Ma (1999) refers to as a "profound understanding of fundamental mathematics" (p. 118) needed for effective mathematics teaching. Nevertheless, these preservice teachers had the impression that the MCK they needed for their lessons was more than adequate. This was in part because they were receiving no feedback to the contrary from their supervising teachers but also partly because they had successfully completed far more sophisticated algebraic work in the advanced mathematics courses required of them in their teaching degree.

The participants' sense of mathematical readiness to teach evident in this study is echoed in two other studies of secondary mathematics preservice teachers (Ball, 1990; Plummer & Peterson, 2009). Hence, a disconnect appears to exist between the MCK needed for advanced mathematics work, which the preservice teachers possess to some degree, and unique aspects of MCK needed for teaching, which they appear to lack. This is despite the former being developed at universities to determine the latter within teacher education programs. Further, while the preservice teachers' completion of advanced mathematics strengthens their knowledge of mathematics, it may also hamper their efforts to hold lower secondary mathematical ideas in a worthwhile form for teaching.

One of the most recent and significant mathematics experiences for the preservice teachers in this study concerned their completion of advanced mathematics courses. Those courses featured topics such as partial differential equations and matrix algebra which are far more mathematically advanced than the lesson content they taught in this study. Consequently, the preservice teachers' MCK for teaching lower secondary algebra was in part, a product of their experiences with more advanced forms of algebra, reflecting von Glasersfeld's (1984) premise that all knowledge is shaped by experience. The often cited meta-analysis of studies conducted in the 1960s and 1970s by Begle (1979, cited in Speer & Hald, 2008) and more recent research by Cooney, Wilson, Albright, and Chauvot (1998) and Kahan, Cooper, and Bethea (2003) provide strong evidence to suggest that success at university level mathematics does not necessarily translate into a strong fundamental understanding of school mathematics for preservice teachers. Further, it is plausible that certain limitations of preservice teachers' MCK such as their lack of verbal precision, their automated treatment of algebraic manipulations, their preoccupation with procedures, and their compression of particular mathematics might be attributed, in part, to their experiences with advanced mathematics courses.

6.2.2.1 Completion of advanced mathematics does not develop precise mathematical language

The preservice teachers in this study chose not to enact "precise and elegant mathematical language" (Lim, 2007, p. 77) which has been associated with effective teaching (Evertson, Emmer, & Brophy, 1980; Good & Grouws, 1977). Instead, the participants' verbal descriptions regularly distorted the mathematics they spoke of, giving their students an impaired version of algebraic content. Not surprisingly, enacting imprecise MCK has been linked with poor student achievement and lesson effectiveness (Land & Smith, 1979; Smith & Cotten, 1980; Smith & Edmonds, 1978), highlighting the negative impact that the preservice teachers' language may have had on the quality of their lessons.

Advanced mathematics courses do not appear to support the development of preservice teachers' verbal facilities with mathematics. Traditionally, the most common instructional format found in tertiary advanced mathematics courses is the lecture, where students listen and take notes (Morrel, 1999; Speer, Smith, & Horvath, 2010). This instructional approach does little to develop preservice teachers' ability to verbalise mathematical ideas in a meaningful way. In addition, Anderson (1980) and Schön (1983) posit that verbal mediation of a task will increase or even disappear as one's performance of that task increases. The preservice teachers' skills in performing the algebraic procedures they taught to their classes would presumably have been honed over many years in their secondary and tertiary mathematics studies, making precise and comprehensive mathematical explanations all the more difficult to cultivate.

6.2.2.2 Completion of advanced mathematics does not develop conceptual knowledge

The preservice teachers' prior mathematical experiences may explain their emphasis on mastery of mathematical procedures and algebraic manipulations in their own lessons. Engelbrecht (2008) argues that students studying mathematics in secondary schools and advanced mathematics at universities learn to perform "predetermined algorithms triggered by key words" (p. 3) which can lead them to view mathematics as "fluency in algebraic manipulations" (p. 4). This view results in what Rasmussen (2001) refers to as "mindless symbolic manipulations" (p. 67) where conceptual knowledge is neither valued by students nor required of them to succeed in the courses. It follows that preservice teachers' experiences with advanced mathematics courses of this kind would lead them to possess and enact few connections to conceptual ideas. This finding is further emphasised by Goulding, Hatch, and Rodd (2003) who found that secondary mathematics preservice teachers viewed their advanced mathematics courses as rote learning, required for success in examinations. It appears that in the rule based learning environment of secondary mathematics and advanced mathematics courses, preservice teachers are not encouraged to develop the conceptual knowledge they need to present robust conceptual connections to their own students.

Preservice teachers need to develop far stronger conceptual knowledge (Conference Board of the Mathematical Sciences, 2001; Steffe, 1990) and connect those understandings with the procedures they present to students. For preservice teachers to do so, they must overcome the procedural emphasis that they have experienced as mathematics students, or better yet, experience a different view of mathematics within those learning experiences. Numerous scholars (Brownell, 1945; Kilpatrick et al., 2001; Tall & Vinner, 1981) claim that knowledge of mathematical concepts is refined and extended as a result of experience. Preservice teachers need mathematical experiences such as those described by Greeno (1978) where attention is drawn to the relevant mathematical concepts of a procedure. Only if preservice teachers are exposed to strong conceptual connections and see the value in those connections are they likely to prioritise and therefore enact strong conceptual knowledge.

The missed opportunity for preservice teachers to revisit secondary mathematical content is a negative consequence of only studying advanced mathematics courses at tertiary level. Ball (1990) and Shoaf (2000) note that taking more advanced maths courses does not give preservice teachers the opportunity to revisit secondary mathematics concepts so their understanding does not really deepen as a result of the additional courses. The findings of this study confirm a lack of conceptual depth held by preservice teachers that appears not to have been developed adequately enough via their completion of advanced mathematics courses.

6.2.2.3 Completion of advanced mathematics leads to compressed MCK

Compressed mathematical knowledge is a natural consequence of learning advanced mathematics. Ball, Lubienski, and Mewborn (2001) explain that as more advanced mathematics is learned, one's own mathematical knowledge becomes more and more compressed. Preservice teachers, having spent years performing simple algebraic procedures such as those taught to students in this study, would have developed mathematical chunks (Miller, 1956) or automated routines, comprising well-practised sets of procedures and sub-procedures. Although automated sets of procedures and sub-procedures can be considered beneficial to possess (Hiebert & Le Fevre, 1986) because they reduce the cognitive load associated with the performance of those routines, they are not ideal for teaching because mathematical details can be missed. Compressed MCK creates a challenge for preservice teachers who must transform their advanced mathematical knowledge into a form that secondary students can comprehend (Shulman, 1986).

The responsibility for assisting preservice teachers to decompress their MCK lies in teacher education. Although the participants' students in this study assisted the preservice teachers to decompress their MCK via mathematical conversations in class, the preservice teachers only scratched the surface of conceptual knowledge and AWOTS when they enacted these knowledge types. Furthermore, school students cannot be expected to become mathematical directors of preservice teachers' mathematics reflection or decompression. It would also be idealistic to expect preservice teachers to self-regulate the decompression of their own mathematical knowledge, given the difficult and unnatural task of finding what is hidden. Decompression is no easy feat, warn Ball and Bass (2000), as preservice teachers must learn "to do something perverse: work backwards from mature and compressed understanding of the content to unpack its constituent elements" (p. 98). Adler and Davis (2006), who noted their participants' compressed and abbreviated mathematics content even in mathematics courses designed specifically for teachers, suggest that advanced mathematics courses don't appear to encourage future teachers to reconfigure their knowledge in a more expanded form. Breaking unproductive mathematical habits formed in the preservice teachers' previous mathematics experiences such as automated, compressed forms of algebraic procedures therefore becomes an important but challenging goal for mathematics teacher educators.

Preservice teachers' students may enjoy a surprising benefit if their teachers are experts in algebraic manipulations, despite their standing as pre-novice teachers. The preservice teachers' lack of attention to the mathematical elements of a compressed mathematical routine actually makes it easier for them to teach, according to Cohen (2011). Cohen (2011) explains that if teachers present knowledge in a compressed form, they reduce the attention required to teach that knowledge. The preservice teachers' delivery of automated, compressed mathematical content may therefore have allowed them to attend to other elements of the classroom context because of the reduced cognitive load associated with performing a mathematical routine. Their ability to respond to contextual elements, described earlier in this chapter, belies their status as pre-novices according to the descriptions of Dreyfus and Dreyfus (1980) who claim that novices rely on context-free rules and context does not guide novices' behaviour. It may be that possessing expert procedural knowledge may have freed up the preservice teachers' cognitive load somewhat and allowed them to respond to student prompted events.

In summary, the preservice teachers' prior experiences in learning mathematics may explain some of the MCK related decisions that preservice teachers make. The lack of emphasis on verbally precise mathematics terminology and conceptual understanding may account for some of the preservice teachers' MCK inadequacies that inform their instructional decisions. The compressed form of their MCK of algebra, as a consequence of performing lower secondary algebraic procedures for many years, may explain their decisions to enact automated routines in their lessons but it may also explain the preservice teachers' ability to respond to certain contextual features of their practicum situation. Although benefits exist for preservice teachers studying advanced mathematics, they also may have contributed to the preservice teachers' poorer MCK related decisions.

6.2.3 Preservice teachers' limited opportunities to develop evidence-based student judgements

Effective secondary mathematics teachers are able to meet the diverse needs of mathematics students (Goos, Stillman, & Vale, 2007). The participants in this study attempted to meet their students' needs by using their judgements about students to inform their MCK related decisions. Their notions of how best to cater for their students in a mathematical realm were often questionable, however, and led them to enact poorer versions of their MCK. Their judgements about students were particularly problematic when they made preactive MCK related decisions because the absence of real students meant they resorted to what they anticipated their students required, mathematically speaking. Successfully identifying mathematics students' needs appears to be a difficult task for preservice teachers, based on the findings of this study.

The poor judgements that preservice teachers hold about students may be explained by a number of factors. Three significant factors, identified in the literature, are explored in this section: (a) the preservice teachers' apprenticeships of observation, (b) the disproportionate number of general education courses compared with mathematics education courses in teacher education programs, and (c) limited and benign practicum experiences.

6.2.3.1 Preservice teachers' apprenticeships of observation

The preservice teachers' weak student judgements may stem from their inexperience in teaching mathematics compared to their years of experience learning mathematics. The majority of pedagogical judgements offered by the preservice teachers in their interviews were phrased as "students need…" rather than "teachers should…". Although they referred to each of the PCK components in the *Mathematical Knowledge for Teaching (MKfT)* framework (Ball et al., 2008), the preservice teachers described the components from the point of view of a student, not a teacher. This subtle difference in expression suggests that preservice teachers may think of themselves as authorities on student learning rather than teaching and view the teaching role predominantly from the perspective of a student.

Mathematics preservice teachers' learning begins with their own experiences as mathematics students which can profoundly impact their teaching careers (Ball, 1988; Cavanagh & Prescott, 2007; Llinares & Krainer, 2006). The preservice teachers' reference to all pedagogical issues through the eyes of a learner is consistent with Lortie's (1975) phenomenon of an "apprenticeship of observation" (p. 61), suggesting that preservice teachers' preconceptions about mathematics teaching are the result of an informal teaching apprenticeship served in this case, over 14 or 15 years as mathematics students. As preservice teachers only view the role of a teacher from a student's perspective, they are unable to fully understand what lies behind teacher actions, such as goals, reflections, and pedagogical frameworks (Lortie, 1975) and that lack of understanding can lead them to make poorer instructional decisions.

It may seem inconceivable to an experienced teacher to knowingly present restricted versions of MCK that are unsuitable for mathematical contexts that lie beyond the current lesson. However, the findings of this study show how the waters can easily become murky for inexperienced preservice teachers. For example, their decisions to enact restricted versions of MCK, described in section 6.1.1, may reflect their attendance to one lesson at a time, as a student might think of mathematics lessons. Their beliefs that students need to achieve success within each lesson and that confusion is a bad outcome appear more consistent with a student's view of a successful lesson (e.g., correct answers with no hurdles encountered), rather than with that of a teacher. An experienced teacher would favour long term understanding over short term success, recognising that moments of student confusion or cognitive conflict are a necessary aspect of learning (Tall, 1991). Lortie's (1975) suggestion that preservice teachers only view the performance of a teacher from the stage, rather than the wings where a more comprehensive view is available, is pertinent for these preservice teachers. Preservice teachers' judgements about what is best for their students appear to be more a product of their time as learners, rather than teachers, and as the student vantage point is limited in the new role of teacher, their judgements are not driving their best decisions regarding MCK.

6.2.3.2 Preservice teachers' general education experiences

Preservice teachers need to develop evidence-based judgements about students in their teacher education program if they are to extend their view of mathematics teaching and learning beyond their student observations of past teachers and lessons. Many scholars contend that preservice teachers must develop pedagogical knowledge, including judgements about students, that specifically relate to mathematics teaching (Ball et al., 2008; Goos, 2013; Graeber, 1999; Kinach, 2002a; Tanisli & Kose, 2013). However, preservice teachers take very few mathematics education classes where PCK might feature. Instead, they are required to complete a number of general education courses (Lawrance & Palmer, 2003) and work out for themselves how general education principles about teaching and learning might apply to the mathematics classroom. The transformation of general education knowledge to mathematics specific knowledge has not been investigated in the literature but according to Stotsky (2006), is unlikely to happen in any subject area, including mathematics. Preservice teachers' university-based opportunities to develop evidence-based judgements about how students learn mathematics and what their mathematical needs might be are limited by the large number of general education courses compared with relatively fewer mathematics education courses.

6.2.3.3 Preservice teachers' practicum experiences

This study concluded that elements of the practicum context can influence preservice teachers' MCK related decisions. In particular, the supervising teacher, the strongest influencing element identified, can offer advice to preservice teachers about the quality of the judgements they hold about students, the suitability of their MCK for teaching, and the quality of their MCK related decisions. The limited opportunities for preservice teachers to teach mathematics lessons during their practicums and the potentially benign nature of the practicum experience reduce the positive influence that the practicum context, and especially that of the supervising teacher, could have on preservice teachers' development of evidence-based student judgements.

The relatively limited number of practicum experiences offered to preservice teachers in Australia (Tatto et al., 2010) may partially explain the participants' underdeveloped judgements about students in this study. Cavanagh and Prescott (2007) claim that knowledge and beliefs about mathematics and mathematics teaching, including student judgements, begin to align with those of the practicum community as preservice teachers participate in practicum experiences. This suggests that with more time in practicum settings, preservice teachers' judgements about students will become more aligned with those of more experienced teachers. However, Australian preservice teachers, in comparison with preservice teachers in other countries, have fewer opportunities to develop student judgements that are based on evidence provided in the practicum context and must instead rely on underdeveloped judgements, based on the limited viewpoint offered by their apprenticeships of observation.

When practicum opportunities are provided, the advice offered by supervising teachers may or may not contribute to the development of the preservice teachers' judgements about mathematics students. The practicum setting can be a benign experience (Lewthwaite & Wiebe, 2012) for preservice teachers where little direction is offered. Alternatively, supervising teachers can offer specific advice but that advice may be unrelated to issues of mathematical content or mathematics specific learning and may instead focus on issues such as behaviour management or classroom routines (Leatham & Peterson, 2010; Markworth et al., 2009).

Supervising teachers may assume that it is not their responsibility to develop preservice teachers' MCK or their judgements about students ((Leatham & Peterson, 2010; Markworth et al., 2009), particularly if they are out-of-field teachers themselves, as was the case for two supervising teachers in this study. At the time that data for this study were collected, the QCT (2006) professional standards against which the participants were judged (Appendix B) did not explicitly reference knowing subject content so supervising teachers were not encouraged or directed to offer advice relating specifically to the mathematics content that preservice teachers should be delivering. Therefore, when preservice teachers are in the practicum setting, the potential for them to develop their judgements about students does exist but without strong direction, there may be only limited development and pre-existing, poor judgements about students may remain.

Notably, the practicum context can lead preservice teachers to develop not only better judgements about students but also stronger MCK. The findings of this study indicate that although MCK development rarely occurred as a direct result of elements of the practicum context (e.g., supervising teacher advice, the term overview, student cohort ability, or the textbook content), given the right practicum circumstances, it is possible. If preservice teachers identify certain mathematical knowledge that students need to be taught in a lesson and they are aware that they do not already possess that knowledge, they will take steps to remedy the situation. In this study, only one preservice teacher did so and the additional knowledge developed was procedural in nature. Nevertheless, the preservice teachers' high regard for the advice offered by their supervising teachers and their desire to meet the requirements of their practicum implies that if other mathematical lapses had been identified, it is likely that the preservice teachers would have responded by developing their MCK to more adequately meet the needs of their students. The practicum context has the potential to positively influence the quality of MCK, should the preservice teachers identify a disparity between the current state of their MCK and the MCK they need to successfully deliver the mathematical content of their practicum lessons. It is the direction offered within the practicum setting that can make the difference.

6.3 Conclusion

This chapter presented the major conclusions of the study and a discussion of theoretical reasons for those conclusions. The chapter began by addressing the second research question, providing conclusions about the visible aspects of enacting MCK. The major findings indicate that preservice teachers pay a disproportionate amount of attention to algebraic procedures in their teaching of algebra lessons. They appear so preoccupied with enacting procedural knowledge that they miss many opportunities to teach connections involving conceptual knowledge, AWOTS, or a wider variety of procedural options. A lack of specialised content knowledge is evident in the preservice teachers' verbally imprecise references to mathematical ideas and a lack of horizon knowledge is evident in the preservice teachers' failure to consider how the restricted forms of MCK they enact will affect their students' understanding of more advanced mathematical topics.

The major findings of research question 1, which pertain to the invisible aspects of enacting MCK, i.e., MCK related decisions, were then presented. The findings were introduced with a model that represented the influencing elements of preservice teachers' decisions that lead to higher and lower quality MCK of algebra. The model and accompanying discussion indicate that the quality of preservice teachers' MCK related decisions depends on the quality of their own MCK, the quality of the judgements they hold about students, the degree of mathematical direction provided to the preservice teachers in the practicum context, and the opportunities they have to revisit mathematics content with students in small instructional settings.

In the latter sections of the chapter, theoretical discussions were provided as possible explanations for the major findings of the study. Literature concerning the situated nature of preservice teacher MCK and preservice teachers' experiences in learning mathematics and mathematics education was used to explain why preservice teachers might hold MCK and judgements about students that are less than ideal for mathematics teaching.

The discussions presented in this chapter suggest that preservice teachers' MCK and the judgements they hold about students need further development. Opportunities exist in both university-based and school-based settings for preservice teachers to develop more robust knowledge of mathematics and students. However, the findings of this study indicate that changes are needed in the kinds of educational experiences that are traditionally offered to preservice teachers. Implications for practice that are based on the major findings presented in this chapter are elaborated in chapter 7.

Chapter 7: Conclusion

7.0 Introduction

High quality secondary mathematics teachers with strong content knowledge are in demand in Australia (Brown, 2009; Hughes & Rubenstien, 2006; Lawrance & Palmer, 2003; McPhan et al., 2008). Strong mathematical content knowledge (MCK) is required for teachers to possess strong pedagogical content knowledge (PCK) (Harel et al., 2008; Thomas, 2003), which has been positively associated with student achievement (Baumert et al., 2010; Hill et al., 2005). Concerns have been raised about the effectiveness of current teacher education programs in producing mathematics teachers with adequate mathematical knowledge (Adler & Davis, 2006; Cooney, 1999; Hsieh et al., 2011; Schmidt et al., 2011; Tatto et al., 2010). Teacher educators, both at universities and in schools, bear the responsibility for providing opportunities for preservice teachers to develop MCK in readiness for their teaching careers. To inform changes to current initial teacher education experiences, teacher educators need to ascertain the quality of the MCK that secondary mathematics preservice teachers present in a live classroom and, as importantly, the factors that affect the quality of that MCK. Hence, this study was a response to the need for a contextualised measure of preservice teachers' enactment of MCK during instruction.

This study explored the experience of secondary preservice teachers as they enacted MCK during practicum lessons on lower secondary algebra topics. Two research questions, each focussing on a different aspect of the preservice teachers' experience were addressed. The first research question explored elements that influenced the MCK that preservice teachers decided to enact in their lessons. The second research question sought the type and quality of MCK enacted and its suitability for algebra teaching. The findings of the research questions were reported in chapters 4 and 5, and were synthesised and discussed in light of pertinent literature in chapter 6. This chapter describes the methodological and empirical contributions the study makes to the field of mathematics teacher education as well as the implications that the study may have for those involved in teacher education practice and research.

7.1 Summary of the study

The preservice teachers enacted a version of the MCK they possessed, a version they believed was suitable to share with lower secondary algebra students. The preservice teachers did not always present all the MCK they held for the algebra lessons they taught, so the MCK that was visible in the lessons represents only the aspects of the preservice teachers' MCK that they chose to share with students. Notably, the less visible aspects of enacting MCK, i.e., the decisions, revealed that the influence that regularly lay behind preservice teachers' decisions to knowingly omit MCK or to restrict the MCK they delivered was the judgements they held about students. The preservice teachers' judgements (which were often questionable) about how they imagined students learn mathematics and what they believed their students needed, mathematically speaking, are likely to be based on their own experiences as students (Ball, 1990; Cavanagh & Prescott, 2007; Holm & Kajander, 2012; Llinares & Krainer, 2006; Lortie, 1975), suggesting that preservice teachers need to develop more reliable, evidence-based judgements about students. Nevertheless, the MCK of algebra that the preservice teachers did teach during instruction offers a solid indication of the types of MCK that preservice teachers hold in more and less robust forms.

In this study, the preservice teachers taught predominantly procedural knowledge, focusing on rules and steps. Their teaching actions featured only sporadic references to conceptual knowledge or algebraic ways of thinking (AWOTS) in their lessons. The presentation of mathematical concepts and AWOTS tended to be superficial in nature and used mainly to support procedural explanations delivered in class. Importantly, the preservice teachers' preoccupation with algebraic procedures and lack of attention to other equally important types of MCK was reflected in their interviews. This suggests that preservice teachers' MCK of concepts and AWOTS may not be as well developed as their procedural knowledge, leading them to deliver lesson content that is lacking in mathematical depth.

The preservice teachers' verbal delivery of mathematical content regularly evidenced inappropriate mathematical language that reduced the quality of their teaching actions. For the majority of the episodes, the preservice teachers' poor language choices were repeated in their episode reflections, indicating they do not possess more precise forms of mathematical language and their MCK is inadequate in this area. The reflections of two participants who did use more precise terminology in their reflection of an episode suggests that preservice teachers may not recognise the pedagogical value of using precise mathematical terminology in their explanations to students.

The live teaching experience and more specifically, talking mathematics with students, provides valuable opportunities for preservice teachers to develop evidence-based student judgements and MCK. The preservice teachers presented higher quality mathematics content when they chose to enact particular MCK in response to one of their students' contributions. When students responded to the content presented in class, the preservice teachers' consideration of their students' questions, comments, and written work led them to form better judgements about their students because those judgements were based on real, rather than hypothetical student needs. The preservice teachers' response to those needs resulted in their re-examining the mathematical content they were delivering, helping them to unpack and then explicitly teach pertinent concepts, ways of thinking, or more specialised forms of procedural knowledge. The opportunity to interact with students in a live exchange of mathematical ideas appears to be an invaluable one for preservice teachers, improving the quality of their MCK and their judgements about what aspects of their MCK are important to share with students.

7.2 Contribution to the field

This study makes methodological and empirical contributions to the field of mathematical education. Methodologically, the study design, comprising an adapted use of stimulated recall and a detailed analytical framework, offers a means by which the connections between decision making elements and enacted mathematical knowledge can be investigated within the classroom context. Previous studies have examined either decision making components (e.g., Leinhardt & Greeno, 1986; Schoenfeld, 2010; Westerman, 1991; Zimmerlin & Nelson, 1999) or specific elements of MCK (e.g., Ball, 1990; Bryan, 1999; Even, 1993; Goos, 2013; Stump, 1999) but not both together, so connections between particular elements of preservice teachers' decisions and MCK had not been explored.

In this study, MCK related teaching actions were partitioned according to pedagogical goals discerned from the preservice teachers' commentaries to create the units of analysis called episodes. The creation of episodes using the goal component of preservice teacher decision making ensured that aspects of enacted MCK could be studied alongside the decisions driving the enactment. The development and implementation of a detailed analytic framework using the literature related to MCK and teacher decision making provided insights regarding the type and quality of MCK that preservice teachers present and associated influencing elements. Hence, this approach makes an important contribution to research methods that aim to connect the process and product of secondary mathematics preservice teachers' MCK related decision making.

Empirically, the study makes a contribution to the body of knowledge pertaining to secondary mathematics preservice teacher decision making. Although the sample of preservice teachers in this study (n = 6) was relatively small, the opportunity to look across multiple preservice teachers and multiple lessons for the same mathematical strand allowed for trends to be discerned between decision making elements and enacted MCK of algebra that have not previously been reported in the literature. The model of influencing elements that appear to positively or negatively impact the quality of MCK delivered in algebra lessons, developed from the findings of this study and described in chapter 6, is a significant contribution. The model provides teacher educators and researchers with an indication of how particular influencing elements can lead to relatively better or worse MCK manifesting in preservice teachers' algebra lessons.

A second empirical contribution which this study makes pertains to describing preservice teachers' MCK. Studies of preservice teachers' MCK reported in the literature have generally been limited by their collection of data outside the context of the live classroom. Although this study was only able to gather MCK related data for a small number of participants, the collection of data from live lessons and using introspective approaches presented a more dynamic perspective on preservice teacher MCK than what is possible in tests and interviews. The preservice teachers' decisions to withhold particular MCK, the presence of conflicting influences behind their MCK related decisions, and the potential for student interactions to help preservice teachers decompress their MCK during instruction highlight the complex nature of knowledge in action for preservice teachers. The MCK that the preservice teachers presented in this study was found to have shaped and been shaped by the context in which it was located (i.e., the practicum and the classroom contexts). The findings affirm the premise of this study that unique insights concerning preservice teachers' mathematical knowledge for teaching can be gained when that MCK is studied within the classroom context.

7.3 Implications for practice

This study has several implications for preservice teacher education. Most notably, a major implication for practice is the need for stronger partnerships to be formed between university-based and school-based educators who work with secondary mathematics preservice teachers. The call for partnership development in Australia is timely, given the recommendations of a 2014 national review of initial teacher education programs include the establishment of stronger partnerships (Teacher Education Ministerial Advisory Group, 2014). In addition, the 2015 conference theme of the Australian Teacher Education Association (2015) focused specifically on partnerships in teacher education. At the university where the participants of this study were undertaking their teacher education program, almost no connections existed between school-based teacher educators and university-based educators of undergraduate (advanced) mathematics, general education, or mathematics education courses. Supervising teachers in practicum schools and university lecturers of advanced mathematics, general education, and mathematics education each have a role to play in helping preservice teachers to develop stronger and more suitable MCK and student judgements for the work of teaching. An approach that has a common vision shared by all educators invested in preservice teacher education would lead to stronger outcomes for secondary mathematics preservice teacher education. That approach would ideally include:

- a reassessment of advanced mathematics course offerings (partnerships between undergraduate mathematics and mathematics education lecturers);
- more opportunities for preservice teachers to develop stronger MCK (partnerships between undergraduate mathematics, general education, and mathematics education lecturers);
- development of preservice teachers' evidence-based judgements about mathematics students (partnerships between mathematics education and general education lecturers);

- stronger mathematical direction within the practicum context (partnerships between university-based and school-based educators).

7.3.1 Reassessment of advanced mathematics offerings

This study suggests that a number of limitations of the MCK that preservice teachers deliver may be associated with their experiences with advanced mathematics courses at university. Current emphases on procedures (Engelbrecht, 2008), "mindless symbolic manipulation" (Rasmussen, 2001, p. 67), and rote learning (Goulding et al., 2003) in university mathematics courses were evident in the preservice teachers' actions in this study. The compressed form of relatively simple procedures, as a natural consequence of having taken advanced mathematics courses (Ball et al., 2001), was also evident in the preservice teachers' lessons. If preservice teachers are to deliver higher quality MCK in their lessons, they need to develop stronger awareness and understanding of mathematical ideas that relate to lower secondary algebra but these ideas are not usually a feature of advanced mathematics courses. Mathematics teacher educators need to work with advanced mathematics lecturers to identify and make explicit within advanced mathematics courses, key ideas, such as mathematical concepts and ways of thinking, that extend preservice teachers' knowledge of the nature of mathematics generally, and more specifically, of lower secondary algebra, beyond mastery of procedures.

7.3.2 Increase opportunities for preservice teachers to develop MCK

The findings of this study indicate that given the opportunity to discuss mathematical ideas with students, preservice teachers re-examine the mathematical content they deliver. This re-examination helps preservice teachers to unpack their MCK, thereby improving the suitability of their own MCK for teaching. This is a phenomenon that should be capitalised upon in teacher education programs. The structure of practicum experiences within teacher education programs, however, limits preservice teachers' experiences of this kind to relatively few days at certain times of the year. Education course lecturers should therefore look to incorporate authentic opportunities for preservice teachers to teach mathematical content to real students outside of the practicum context, as students unknowingly assist preservice teachers to reflect and refine their own mathematical understandings for teaching as interactions take place. Teachers of advanced mathematics and mathematics education courses also need to investigate other kinds of learning experiences that allow preservice teachers to revisit secondary content that they will be teaching from a more advanced perspective, which they may not have considered since their own high school classes (Kahan et al., 2003). The introduction of a course intended for just this purpose has been established at the university where this study took place in response to the preliminary findings of this study. The development of similar university courses (e.g., Artzt, Sultan, Curcio, & Gurl, 2012; Loe & Rezak, 2006) and course materials (e.g., Sultan & Artzt, 2011; Usiskin, Peressini, Marchisotto, & Stanley, 2003) in universities around the world further evidence how preservice teachers might better develop their MCK in teacher education programs.

Preservice teachers encounter difficulties in expressing mathematical ideas with precision and need opportunities to form more suitable explanations of mathematical ideas. The findings of this study showed that preservice teachers present mathematical ideas in a range of verbally imprecise ways, which contrasts with their more precise forms of written mathematics. The preservice teachers' lack of experience in communicating mathematics verbally and their expertise in algebraic manipulations, which leads to compressed forms of algebraic knowledge, make detailed descriptions difficult for them to form. The relatively small proportion of time spent in practicum settings in teacher education programs means that preservice teachers need opportunities to develop their verbal explanations in university settings as well as school settings. Hence, more verbal articulation of mathematics ideas needs to be incorporated in advanced mathematics courses (e.g., open discussions of solution paths), general education courses (e.g., in class teaching opportunities), and mathematics education courses (e.g., explicit attention paid to developing appropriate mathematical terminology for secondary teaching). Opportunities such as these can help preservice teachers learn how to phrase mathematical ideas more clearly and accurately.

7.3.3 Develop preservice teachers' judgements about mathematics students

This study clearly showed that the judgements preservice teachers had about mathematics students and mathematics learning impacted the MCK they were prepared to deliver. The quality of MCK that manifests in preservice teachers' lessons is therefore a function of their MCK but also of their PCK or more specifically, the knowledge and beliefs they hold about mathematics students. This study showed that preservice teachers hold preconceived views about students that can lead them to teach poorer versions of the MCK they possess. However, when those judgements are refined in the moment, via live student interactions, preservice teachers tend to present broader, deeper, and more connected MCK. These findings highlight the need for preservice teachers to reflect on and address the beliefs they hold about how students learn algebra in lower secondary school and to develop more evidence-based knowledge of secondary mathematics students. General education courses, such as educational psychology, and mathematics education courses are those that may contribute most explicitly to preservice teachers' development of PCK, which includes their judgements about students and the learning of mathematics. Within mathematics education courses, opportunities to engage with relevant mathematics education literature and experience teaching encounters with real students may assist preservice teachers to form more realistic judgements about the students they teach. Partnerships between mathematics education and general education lecturers are also needed to ensure that preservice teachers are developing evidence-based knowledge of mathematics students in their general education courses. As preservice teachers become exposed to more and more aspects of mathematics teachers' work, it is envisaged that their evidence-based knowledge of how mathematics students learn might be further strengthened and higher quality MCK in the classroom may follow.

7.3.4 Stronger mathematical direction in the practicum context

This study concluded that elements of the practicum context, and particularly the supervising teacher and the term overview, strongly influenced preservice teachers' MCK related decisions. Yet, the brevity of the term overview content statements and the relative freedom provided by supervising teachers when preservice teachers planned their lessons resulted in little specific direction where mathematical content was concerned. Supervising teachers can assist preservice teachers to prepare lessons with more mathematical rigour by alerting them to the importance of conceptual understanding, ways of thinking, mathematical connections, and the delivery of content that maintains mathematical integrity. Preservice teachers appear to hold their supervising teachers in very high regard, so specific advice of this kind may encourage preservice teachers to re-examine the content they are planning to teach from a different viewpoint and take steps to improve the quality of the MCK they deliver. Supervising teachers might also encourage preservice teachers to use pedagogical approaches such as group work which include interactions with smaller groups of students. The promotion of smaller instructional settings by supervising teachers can benefit preservice teachers and their students because the better versions of preservice teacher MCK tend to be enacted in those settings.

If more specific mathematical direction is to be offered, supervising teachers with considerable MCK themselves are needed to ensure that preservice teachers receive high quality advice. Using the *National Professional Standards for Teachers,* produced by the Australian Institute for Teaching and School Leadership (AITSL, 2014), as a guide, Australian teachers certified as “highly accomplished” or “lead” teachers would have met teaching standards that include offering advice regarding subject content. It is recommended that suitable supervising teachers be identified and provided with training to ensure they have the skills needed to support preservice teachers during their practicums (Teacher Education Ministerial Advisory Group, 2014). Ideally, all preservice teachers would be mentored by teachers of this calibre but this is not always the case.

The shortage of trained mathematics teachers in Australian secondary schools (Brown, 2009; Lawrance & Palmer, 2003; Sullivan, 2011) and the imminent retirement of many experienced mathematics teachers (Harris & Jensz, 2006) have negative consequences for preservice teacher education. Preservice teachers may be supervised by out-of-field mathematics teachers, who continue to teach approximately 40% or more of lower secondary mathematics classes in Australia (McKenzie et al., 2011; Thomas, 2001), rather than qualified, or ideally, qualified and highly experienced secondary mathematics teachers. Partnerships between university and school-based educators would be invaluable if the supervising teachers were out-of-field teachers themselves, which was the case for two of the supervising teachers in this study. If the AITSL national teaching standards are operationalised in the criteria by which Australian preservice teachers are judged on their practicums, content knowledge may well be given greater emphasis and out-of-field teachers may in turn need support in this area. Hence, collaborations between university-based and school-based educators, as recommended by Leatham and Peterson (2010) and the Teacher Education Ministerial Advisory Group (2014), are needed to support preservice teachers’ development of their MCK before, during, and after their practicum experiences.

7.4 Directions for further research

This study highlighted the unique and complex nature of MCK related decision making and teaching actions by preservice teachers. The study findings indicate that studying MCK and decision making in the classroom is a fruitful line of investigation, inviting researchers to further explore this phenomenon. The preservice teachers in this study were perhaps more adept at performing algebraic procedures, given their history of advanced mathematics, so a study of MCK decision making that involves a mathematical strand in which the preservice teachers are less confident might produce different trends. More studies are needed to gauge more comprehensively the MCK that preservice teachers choose to present in practicum lessons which would, in turn, offer teacher educators better indications of the MCK that preservice teachers tend to deliver during instruction.

The findings of this study are not generalizable to all secondary preservice teachers. The participants of this study were all located at one tertiary institution, were similar in age, and received their secondary mathematics education in one Australian state, so many aspects of their mathematics and education experiences would have been very similar. It would therefore be beneficial to investigate the MCK related thoughts and actions of preservice teachers with different secondary mathematics teacher education experiences to supplement the findings of this study.

The study findings were limited to decision making influences that were identified primarily by the preservice teachers. The study indicated that the preservice teachers' beliefs about students impacted their MCK related decisions, but it could not ascertain the impact of other beliefs that preservice teachers may have held but did not share. While some beliefs about the nature of mathematics, for example, could be inferred, the participants in this study did not identify them, let alone use such beliefs as justifications for their MCK related decisions. Future studies could potentially explore more deeply the beliefs held by preservice teachers and their impact on preservice teachers' MCK related decisions and actions.

Developing preservice teachers' MCK for teaching is a worthy goal for teacher educators. Hence, it is suggested that future studies explore how secondary preservice teacher MCK can be further developed in practicum and university settings. Within the school setting, longitudinal studies that focus specifically on preservice teachers' MCK are needed. This study identified potential areas of MCK growth for preservice teachers within just one or two practicum lessons but it provided only a snapshot of the preservice teachers' MCK within a practicum phase. It would be valuable to gauge how a secondary preservice teacher's MCK develops over the course of a three or four week practicum phase (and over the period of a four year degree) and what additional elements, if any, influence the quality of MCK taught during longer practicum experiences. This could contribute to the generation of a "pre-novice to expert" description of MCK so that preservice teachers' MCK development can be gauged. Within the university setting, learning experiences that lead to preservice teachers re-examining their own MCK need to be trialled and evaluated, to ascertain the types of experiences outside of the practicum setting that lead to higher quality MCK enactment within the practicum setting.

7.5 Final conclusion

The aim of this study was to explore the type and quality of MCK that preservice teachers enact during practicum algebra lessons and the influencing elements that impact the MCK related decisions behind their actions. The study showed that the preservice teachers purposefully delivered aspects of mathematical content in pursuit of pedagogical goals. The preservice teachers' efforts to deliver appropriate algebraic content were impacted by different influencing elements, which led them to teach MCK of varying mathematical quality. Their attempts were hampered by limitations of their own MCK, which were characterised by a dependence on automated procedures, an absence or compression of conceptual knowledge and AWOTS, and a distinct lack of verbal precision.

It has been established in the literature that teachers' MCK informs their PCK (Baumert et al., 2010; Harel et al., 2008; Shulman, 1986) and this study showed that the reverse is also true. Preservice teachers' PCK, and more specifically, their judgements about mathematics students, inform their decisions about the MCK that is most suitable to enact in a live algebra lesson. Underdeveloped judgements regarding what preservice teachers imagined their students needed tended to lead them to deliver poorer versions of the MCK they did possess. In contrast, when live interactions with students led to spontaneous MCK related decisions, the interactions highlighted students' real mathematical needs, prompting the preservice teachers to rethink their judgements about students, unpack their MCK, and consequently respond by presenting better quality MCK. The preservice teachers' competence in other teacher knowledges and the situations in which they find themselves in a live lesson can impact the MCK that is evident in their teaching.

The preservice teachers in this study sought to plan and implement the best quality mathematics lessons they could. In some ways, teacher educators are like the participants of this study. With the very best of intentions, teacher educators draw on the knowledge available to them when they themselves make instructional decisions about their own students (i.e., preservice teachers). The findings reported in this study have the potential to strengthen the evidence-based knowledge that teacher educators have about the needs of secondary mathematics preservice teachers. It is the responsibility of teacher educators to make the changes needed to current teacher education practices that will allow preservice teachers to develop high quality mathematical content knowledge that can lead to stronger pedagogical content knowledge, and ultimately, better student achievement.

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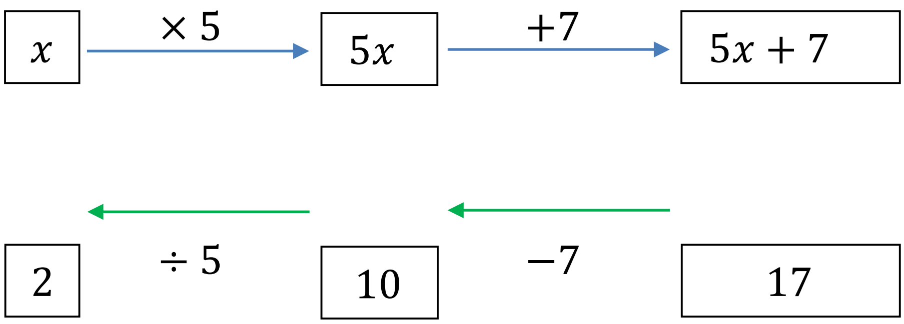
Appendix A: Methods to solve linear equations

Method 1. Backtracking method used to solve $5x + 7 = 17$



Two is the number in the box beneath x, so $x = 2$.

Method 2. Balance method used to solve $5x + 7 = 17$

$$5x + 7 = 17$$

$$5x \cancel{+7} \cancel{-7} = 17 - 7$$

$$5x = 10$$

$$\frac{\cancel{5}x}{\cancel{5}} = \frac{10}{5}$$

$$x = 2$$

Method 3. Transposing method used to solve $5x + 7 = 17$

$$5x + 7 = 17$$

$$5x + 7 = 17 - 7$$

$$5x = 10$$

$$5x = \frac{10}{5}$$

$$x = 2$$

Appendix B: QCT - Professional Standards for Queensland Teachers

1. Design and implement engaging and flexible learning experiences for individuals and groups.
2. Design and implement learning experiences that develop language, literacy, and numeracy.
3. Design and implement intellectually challenging learning experiences.
4. Design and implement learning experiences that value diversity.
5. Assess and report constructively on student learning.
6. Support personal development and participation in society.
7. Create and maintain safe and supportive learning environments.
8. Foster positive and productive relationships with families and the community.
9. Contribute effectively to professional teams.
10. Commit to reflective practice and ongoing professional renewal.

(QCT, 2006)

Appendix C: AITSL – Australian Professional Standards for Teachers

1. Know students and how they learn.
2. Know the content and how to teach it.
3. Plan for and implement effective teaching and learning.
4. Create and maintain supportive and safe learning environments.
5. Assess, provide feedback, and report on student learning.
6. Engage in professional learning.
7. Engage professionally with colleagues, parents/carers, and the community.

(AITSL, 2014)

Appendix D: Participants' pedagogical approaches

Pedagogical approach		Kate, lesson 1	Kate, lesson 2	Grace, lesson 1	Grace, lesson 2	William, lesson 1	Sam, lesson 1	Sam, lesson 2	Ben, lesson 1	Thomas, lesson 1	Thomas, lesson 2
Direct instruction strategy (Killen, 2013; Lasley et al., 2002)	Pre-requisite learning	✓	✓	✓	✓	✓	✓		✓	✓	✓
	Presenting new material*	✓		✓		✓	✓	✓		✓	✓
	Independent practice	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Problem solving strategy (Killen, 2013)	Non-routine problems					✓					✓

* *Presentation included modelling of procedures, guided practice with corrective feedback, and/or note taking.*

Appendix E: Lesson observation template

Lesson observation recording sheet 1 [Complete in real time during the class observation]

Preservice teacher:__________________________________ Lesson number: _______ Year level and topic: _____________________________

Segment type (refer to lesson plan)	**Instructional setting**	**Reason segment occurred**	**Class resources**	**Mathematical terminology**	**Hotspots**
I – introduce maths *C – consolidate maths* *D – develop maths* *R – review maths*	*WC – whole class* *WCL – whole class listening to 1 student and teacher* *SG – small group* *I – individual*	*LP – following lesson plan* *Or* *Events such as:* *SQ – student asks a question* *SC – student makes a comment* *SE – student makes an error* *CC - Class look confused* *ST - Supervising teacher intervenes*	*Calc – Calculators* *Com - Computer* *W – whiteboard* *Ppt – ppt slides* *SB – smart board* *M – manipulatives* *T – textbook* *H - handout*	*Write teacher's written or verbal maths language here*	*Things I want to know more about (Typical and atypical events - errors, problems, interesting/unusual happenings, etc.)*

Appendix F: Contextual information template

Lesson observation recording sheet 2

Preservice teacher: ______________________________________

Lesson Background [Complete on the day of the class observation (or follow up in interview if necessary]

Date: _________________ Term: ________ Week number (of the term): ___________

Lesson number (of the day): ________ Maths lesson number (of the week) : _______

Starting time of lesson: _______________ Finishing time of lesson: _______________

How the maths content of the lesson is situated:

This lesson forms part of the unit ______________________________

which is _________________________long.

The timing of this lesson within the unit is ____________________________________

Lesson context [Complete at some point during the class observation]

Adults in the room: __________ Students in the room: _____ girls_____ boys

Equipment and resources in the room (may not be used, but are there):

____

____

General impressions: The class

The room : ____

The mood (general) of the students: _______________________________________

The behaviour of the students: ____

The attitude of the students (to maths): ____

Level of maths talk in the class: ____

Influence of supervising teacher/teacher aides etc.: ______________________________

Other impressions: ____

General impressions: The preservice teacher

Mood of the preservice teacher:____

Attitude of the preservice teacher (to taking this class): ___________________________

Comfort with teaching content: ____

Other impressions: ____

Appendix G: Interview protocol

Initial statement

Thanks for meeting with me today and letting me watch your lesson. Today I'm not here to critique your teaching in any way. For me, today is all about learning as much as I can about how you used your own maths knowledge while you were teaching. This is so I can be as accurate as I can in my descriptions of how the lesson unfolded. I'll start today by asking you a bit about the lesson in general and then we'll have a look at some video excerpts and talk about those. OK? Any questions for me at this stage?

General lesson discussion

Q1: Let's have a look at the lesson outcomes (refer to lesson plan and point them out). As far as [insert particular maths outcome] is concerned, how did you feel you went with getting that/those outcomes across?

Q2. Can you tell me how you put the lesson together?

[prompt for supervising teacher influence]

[prompt for developing MCK in any way – extra prep?]

[prompt for prior lessons taught, where this lesson is situated in unit]

[collect artefacts – unit plan, textbook chapter, resources]

Stimulus recall

Q3.Now for the fun part! I am going to play you some parts of your lesson and I'd like you to tell me as you watch what you remember thinking at the time. I'll start the video and you can pause it anytime [show them how on the laptop] and tell me what was happening at that time. Any questions?

[Begin playing footage]

As footage plays

- If the preservice teacher pauses the footage and talks about MCK related actions, do not intervene.
- If the preservice teacher does not pause the footage and talk, comment on visuals (I notice you did/said/pointed out…).
- If participant still does not respond in detail, ask questions like (prepare questions for incidents in each video excerpt]:
    - "Can you tell me about that?"
    - "What was your reasoning for…?"
    - "What made you decide to …?"
    - "What was happening there?"
    - "Why was that important to you?"
    - "Mathematically, why do you reckon that kids might have struggled/found this hard/were confused?"
    - "How did you know that?"
    - What would you have liked the student/s to say/write?

After stimulus recall prompts

Q4. Now that you've taught [insert topic] and you've had the chance to look back at the video, mathematically speaking what are your thoughts now?

Thank you for your time.

Appendix H: Preservice teacher background information

1. Age ___________
2. In what state/school did you go to high school? ___________________________
3. What maths subjects did you take at school (senior)? ______________________
4. What kind of results did you get for maths in junior? __________

   Senior maths? _____________
5. University maths results? _____________________________
6. Any other courses in maths taken? If yes, details: ________________________
7. How many maths lessons have you taught on your previous practicums (topics, grades – whatever details can be remembered):
   - 1st year (1 week): ____
   - 2nd year (1 day/f'night for 10 weeks): ______________________________
   - 2nd year (2 weeks, July): ____
   - 3rd year (1 week, April): ____
   - 3rd year (3 weeks, July): ____
   - 4th year (2 weeks, April): ____
8. Have you had any other maths teaching experience? If so, what type and for how long?
   - Tutoring:

     ____
   - Other: ____

Thank you for your time and participation!

Appendix I: NVivo coding scheme

Nodes

Name	Sources	References
1 General lesson commentary	10	26
2 Episode data	10	378
3 Enacted MCK_What MCK did the participant enact	10	1035
1 CCK enacted in episode	10	349
Knowledge of awots	10	109
Knowledge of concepts	10	42
Knowledge of procedures	10	198
2 SCK enacted in episode	10	104
3 HK enacted in episode	1	1
4 Limitations in episode	10	147
5 Reason for MCK limitation (if one is given)	10	26
6 Was there a missed opportunity in episode	10	154
7 Possible reason for missed opportunity	10	117
8 Consistency between MCK in episode and interview	10	137
4 Enacted MCK_What was the participant thinking	10	543
1 - Intended goal or goals	10	216
2 - Extra explanations	10	266
3 - Tensions between MCK and other factors	9	61
5 Omitted MCK_What MCK did the participant withhold	7	85
1 CCK	7	25
1a SCK	6	20
2 Was there a missed opportunity because MCK was omitted	6	19
3 Possible reason for missed opportunity (omitted MCK)	6	21
6 Omitted MCK_What was the participant thinking	7	33
Discarded goals for omitted MCK	7	25
Extra explanations	3	8

Appendix J: Email to preservice teachers

Hi [preservice teacher],

Earlier this year, you were asked to complete a consent form regarding participation in a research project with me (for my PhD). This week, I opened up the sealed envelope and was so pleased to see that you have agreed to let me visit you during your practicum. First of all, thank you! It is wonderful to have you as part of the project.

I would like to make two times with you to observe you teach a junior maths class during the week of [23-27 July (your third week)] if this works for you and your supervising teacher. Within a day or two of your teaching, I would also like to come back to speak to you individually in a stimulated recall interview (I show you excerpts of your lesson to better understand what you did and why you did it). You name the time and place for us to meet.

Your school has approved my visit and knows that I will be visiting at some point. I have made it clear that I am not there to mark you in any way.

I know that you can't confirm any days or times yet but just wanted to touch base with you for now. Could you please send me a quick email back letting me know two things:

1. Could you please confirm that you are still happy for me to come and observe you teach twice/ speak to you about it the following day?

2. What are your preferred contact details (to confirm days and times for me to visit you)?

Thanks again for being a part of the project and I look forward to hearing from you soon.

Kind regards,

Leah

Appendix K: Email to school principals

Good morning [name],

My name is Leah Daniel and I am a high school mathematics teacher who is undertaking doctoral studies at JCU in how preservice teachers teach mathematics. The research requires me to video preservice teachers teaching maths in schools. The purpose of this study is to improve teacher training courses.

One of these preservice teachers, [preservice teacher name], will be completing his/her professional experience at your school in July. [Preservice teacher] has consented to be a part of this study. I am writing to ask for your permission to videotape [preservice teacher] teaching two lessons in a junior mathematics class during his/her time at your school.

I have attached an information sheet to provide you with details about the study and your school's potential involvement. The contact details for myself and my supervisors are also included should you wish further information.

As a parent and a teacher myself, I understand the importance of protecting your students' privacy and ensuring that my presence in the school provides no disruption to your students or teachers. The information sheet outlines the steps that will be taken to ensure that this occurs.

Please let me know what you think of this proposed course of action. Thank you for reading about my research project. I look forward to hearing from you or the appropriate contact person in your school soon.

Kind regards,

Leah Daniel

Appendix L: Email to supervising teachers

Hi [name],

My name is Leah Daniel and I am a high school mathematics teacher who is undertaking doctoral studies at [University] in how preservice teachers teach mathematics. The research requires me to video preservice teachers teaching maths in schools. The purpose of this study is to improve teacher training courses.

One of these preservice teachers, [preservice teacher name], is currently completing his/her professional experience with you. [Preservice teacher] has consented to be a part of this study and your school has given me permission to videotape [preservice teacher] while at your school. Although [preservice teacher] has probably spoken to you about my visit, I wanted to introduce myself and to confirm the possible days and times that I will be observing [preservice teacher]. The two lessons that I am planning to observe are:

1. [Tuesday, 23rd July at 8.50am]

2. [Thursday, 25th July at 11.45am]

Would these lesson times be suitable for you?

As a parent and a teacher myself, I understand the importance of protecting your students' privacy and ensuring that my presence in the school provides no disruption to you or your students. The information sheet attached outlines the steps that will be taken to ensure that this occurs and provides more details about the study. The contact details for myself and my supervisors are also included should you wish further information.

Please let me know what you think of this proposed course of action. Thank you for reading about my research project. I look forward to hearing from you soon.

Kind regards,

Leah Daniel

Appendix M: Practicum school information sheet

Background of the project:

Mathematical content knowledge (MCK) for teachers is a particular type of applied mathematics where mathematics is understood in a different way from how it might be understood by other professionals. Research has shown that MCK helps teachers teach mathematics effectively to their learners. This research project aims to find better ways of developing the mathematical content knowledge (MCK) of preservice mathematics teachers to better prepare them to be effective secondary mathematics teachers. The study is being conducted by Leah Daniel and will contribute to a Doctor of Philosophy in Education at James Cook University. The study requires Leah to video preservice teachers teaching maths in schools. The purpose of this study is to improve teacher training courses and findings from the study will be used in research publications and conferences.

Involvement of your school:

Secondary mathematics preservice teachers who undertake their professional experience at your school, with your permission, will be video recorded while teaching two lessons in a junior secondary mathematics class. The focus of this study is solely on the preservice teacher and not your students in any way. In order to protect the identity of all students in the classroom at the time of videotaping, the video camera will be positioned at the back of the room so that only the backs of some of the students' heads are visible. In the unlikely event that a child's image is captured on video as a result of a child moving around the room, the image will be deleted from the video footage.

The school will not be identified in any way in the study.

Appendix N: Information letter to parents

Dear parents/carers,

My name is Leah Daniel and I am a high school maths teacher who is undertaking doctoral studies at JCU in how trainee teachers teach maths. The research requires me to video trainee teachers teaching maths in schools. The purpose of this study is to improve teacher training courses.

One of these trainee teachers is working with your child's maths teacher. The trainee teacher will be videotaped for two lessons over the next three weeks.

The focus of this study is solely on the JCU trainee teacher and not on your child in any way. The video camera will be positioned at the back of the classroom. This means that the back of your child's head may be in the video.

I am writing to you in the unlikely event that your child's image is captured on video if your child moves around the room. Should that be the case, the image will be deleted from the video footage. The project concerns the trainee teacher and not the students. A teacher from [name of school] will check that the footage does not contain any student images before the footage is taken from the school.

The school will not be identified in any way in the study.

If you have any questions about the study, please contact me or my supervisor, Dr Jo Balatti.

Yours faithfully,

Leah Daniel

Appendix O: William's *Doing the opposite* episode

CONTEXT

Who was involved: William (preservice teacher), his students Lachlan and Maria, and the remainder of his Year 8 class

Location of the episode: This was a learning activity implemented by William in the last lesson of a unit on Algebra that introduced the topic of solving equations. To William's knowledge, students had not been exposed to solving equations before.

Directly before this episode: William verbally gives the following scenario, "I'm thinking of a number, and when I add six, I get thirteen. Can you tell me what my number is?"

Lesson episode:

William What did somebody get for the question?

Lachlan Seven.

William Seven, who else got seven? Keep your hand up if you got seven as well.

Students [Many students raise their hands]

William Alright. Well, seven was the number I was thinking of but how do we know it was seven?

Episode reflection 1 (interview):

I wanted them to realise that to get the answer they had to do the opposite. So if I was adding the number, to go backwards, to work backwards from the answer to the unknown, they had to do the opposite.

Lachlan I subtracted it.

William You subtracted it. But when I said my sentence, I said, "When I *added* six" so why did you *subtract* from the answer? Yes, Maria?

Maria I just plussed the six on.

William OK, so you thought in your head, "What plus six equals thirteen." [nods at Maria to confirm he has understood her correctly]. Yep [nods again to confirm she agrees with his explanation]. Okay, Lachlan, you were saying that you subtracted something. Can you…

Lachlan No, you said it was thirteen.

William Yeah. Can you explain to the class what you were thinking in your head when you were trying to work out my number.

Lachlan Well you said the number's thirteen. That's what it equals. And that it was plus. And so I just minused it and so I worked back and back and back with six until I got to the answer.

William Yep.

Lachlan So I got three and then four, five, and six.

William cleans the board and writes up the equation, $a + 6 = 13$.

William So I said, "My number, when I add six, the answer is thirteen." And when Maria worked it out, she said in her head, "Ok, we have something, we add six, we get thirteen, what is it?" Lachlan was talking about subtracting something. What were you subtracting Lachlan?

Episode reflection 2 (interview):

Yeah, I kept coming back to Lachlan, 'cause Lachlan had specifically said, "I subtracted whatever the number was," and I was trying to get them to connect that the subtraction is the opposite of addition.

Lachlan Oh, the six

William You were subtracting six, away from what?

Lachlan Thirteen

William To get the answer. So you were doing the opposite to what I was doing, weren't you?

Lachlan Yup.

William OK. So that's working backwards.

Appendix P: Sam's *Backtracking* episode

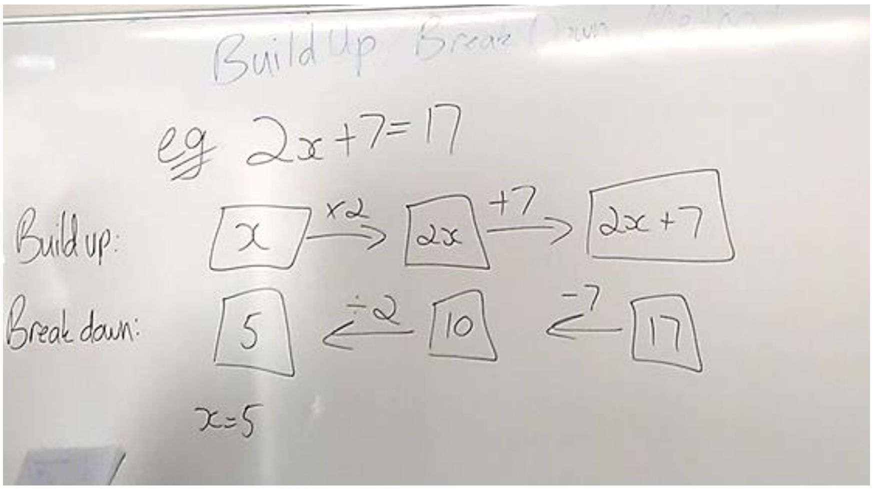
CONTEXT

Who was involved: Sam (preservice teacher) and his Year 8 class

Location of the episode: This was a learning activity implemented by Sam in the middle of a unit on Algebra, to introduce the topic of solving equations by backtracking. Students had previously been exposed to solving equations using the transposing and balance methods in the unit.

Directly before this episode: None. This episode occurred at the start of the lesson.

Board work developed during episode:



Lesson episode:

Sam All right. How this works is we look at first what's happening to x, so we start with x and we write a little box around it. Now, what's the first thing that's happening to x? So using, remember your BOMDAS. Does everyone remember your BOMDAS? Order of operations?

Students: [Many call out at once words including: brackets, multiply, divide, addition, and subtraction]

Episode reflection 1 (interview):

Researcher: *Why is BOMDAS so important for the method?*

Sam: *Because that's when I say, "What's happening to x first?" Then we're multiplying first. We're not adding. That's how it makes sense. What happens to x first. Because for the build up part of it, that's all we're concerned about - what's happening to x and the order in which it's happening. So the first thing that's happening to x is we're multiplying by two. It's really important that we have the order. And then seven is getting added to it.*

Sam: Addition and subtraction. All right very good. So, what's the first thing that's happening to x using BOMDAS here? Can someone tell me? Yup?

Student: Multiply by two.

Sam: Multiplying by two. So using BOMDAS, multiplication comes before addition. So multiplying by two is the first thing that's happening. And that gives us two x [adds first arrow and "$2x$" to the first row of working]. This is the buildup stage [writes the words 'build up' at the top of the board].

Sam: Now, what's happening next to x? After we multiply by two, what happens?

Student: Plus seven?

Sam: You add seven to it. And that gives us our whole left side. Two x plus seven. Now we also know that two x plus seven is equal to 17. So we start with this side and we work backwards.

Student: Do we have to do that [gesturing to Sam's board work]?

Sam: Yup. Straight under like that. This is our breakdown [Writes "break down" to the top of the board]. So whatever you've done to get to two x plus seven, we have to do the opposite to what that equals, 17. So working backwards. We added seven.

Now we take away seven, to give us 10. Same here. We multiplied by two. This time we divide by two.

Episode reflection 2 (interview):

Researcher: You mentioned earlier that you don't see this method as something they'll use down the track.

Sam: Forever, yeah.

Researcher: What value did it hold for you then?

Sam: Mainly so they could see that we're using inverse order of operations now. And just another way, I guess, if they aren't quite getting it this way now, they can use it until … because it's going to help them work out how to do it this way.

Appendix Q: Worded questions

Worded questions requiring dual treatment of equals symbol and equation

No. of equations	Equation as process	Equation as object
Single linear equation	1. A number is doubled to obtain the result 32.	3. Emily's age in 10 years time will be triple her current age. She is currently E years old.
Simultaneous linear equations	2. Find two numbers whose difference is 5 and whose sum is 11.	4. One number is 5 less than three times a second number. If the first number plus twice the second number is 15, find the two numbers.

Appendix R: William's *What's the trick* episode

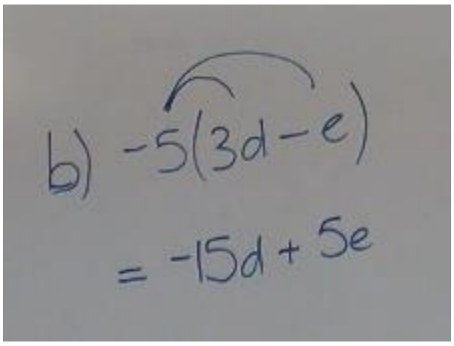
CONTEXT

Who was involved: William (preservice teacher) and his Year 8 class

Location of the episode: This was a learning activity implemented by William in the last lesson of a unit on Algebra, to review manipulation of algebraic expressions to remove the brackets. Students had been taught the procedure earlier in the same week.

Directly before this episode: The expression, $2(a+b)+4(a+b)$ was discussed and rewritten without brackets.

Board work developed during episode:



Lesson episode:

William: Looking at the second one. What's the trick with this one here?

Student: Minus

William: Negative out the front. So do we have to do anything different?

Student: Nuh

William: Nuh. Exactly the same. We're still doing multiply, it's negative five *multiplied* by everything inside the brackets. We just need to make sure we keep the little negative sign with it, with the five. So what's the first step? The first step would be to multiply through, everything outside times everything inside. So if we multiply negative five…

Student: Negative 15 d

William: Yep, so we're doing a negative times a positive, what answer are we gonna, what type of answer, Lachlan?

Lachlan: Negative times a…

William: Negative. So five, negative five times 3 d, what are we gonna be left with?

Lachlan: Oh, fifteen, yeah fifteen d. *Negative* fifteen d

William: Negative 15 d, yep, that's right

William: So now looking again, we're multiplying negative five times negative e. So if we're timesing two negative numbers, what type of answer are we going to get?

Student: A positive one.

William: A positive answer. It means we can say it's a plus sign here [adds + to the second line of working on board] 'Cause it's positive. Now we're doing negative five times negative e. What are we gonna be left with?

Student: Five

William: Five or maybe something else.

Students: Negative five

Lachlan: e, five e.

William: Alright, we're saying it's positive, remember. It's negative five times negative e

Student: So it makes a positive.

William: So it makes a positive.

Episode reflection (interview):

William: Mmm. Even today they were still getting stuck on,..., I think the main problem they have is [points to screen] where it's got the 'take e' there, they see it as a subtract and not as a "plus a negative." That's where a lot of the kids get stuck. I get stuck explaining that to them because…yeah.

Researcher: Did you phrase it like that to them?

William: Not really, cause a lot of 'em… Oh, we kind of, the way I phrased it always has been that the sign has to like stick with it. The sign, the one that's in front of it is, for the sign… So yep, I haven't tried phrasing it that way actually [chuckles].

Researcher: Well that's interesting that the way you see it is not…

William: …is not the way I speak about it, yeah. Yeah…

William: What's the number, what's the answer?

Students: Five e.

William: Five e. Remember we need to take both of them, the five and the e, so it's not positive five or positive e, it's positive five e. So we're gonna say plus five e [adds $5e$ to the second line of working on the board]. Now do we have any like terms that we can simplify here? Nuh, so that's our final answer.